Composition Laws

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ECC 2010

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- suggest several open problems in computational number theory (and algebraic geometry)

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- discrimiant $b^2 - 4ac$ is the discriminant of the corresponding quadratic ring

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- $b^2 - 4ac$ is the branch locus of the map from C to $\mathbb{A}^1$

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- when $R = k[x_1, \ldots, x_n]$ for a field k (Quillen–Suslin theorem)

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- The C on the right correspond, geometrically, to surfaces with degree 2 maps to the plane $\mathbb{A}^2$ over $\mathbb{F}_q$.

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- taking (a, b) over $\mathbb{F}_q[t]$ gives Mumford representation of points on the Jacobian of a hyperelliptic curve

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- over each R the best method for computation of the composition might differ
- for each R, the reduction theory to find a unique representative in equivalence classes of forms is a potentially new problem, both theoretically and algorithmically

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Main interesting aspects: reduction theory, efficient implementation

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- compute this class group once, and then compute class groups of many quadratic extensions of R

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- as with binary quadratic forms, there is a version of this theorem over any ring (variety, scheme...)

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- understand reduction theory (even with one n, and one base ring)