

# Composition Laws

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- see how they are pieces of a larger story
- suggest several open problems in computational number theory (and algebraic geometry)

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- reduction theory to find a unique reduced representative of each  $GL_2(\mathbb{Z})$  class
- discriminant  $b^2 - 4ac$  is the discriminant of the corresponding quadratic ring

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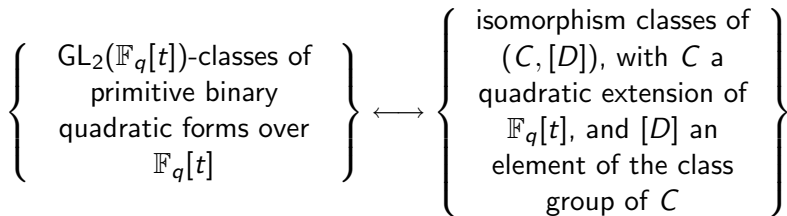
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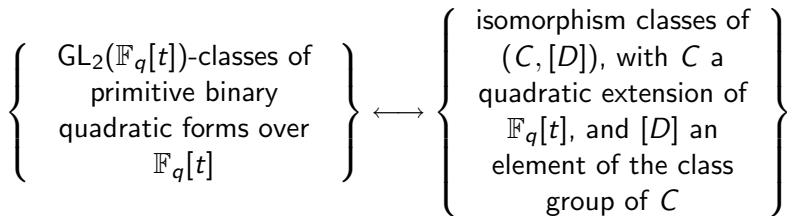
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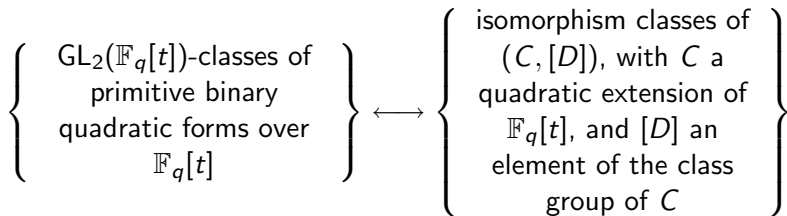


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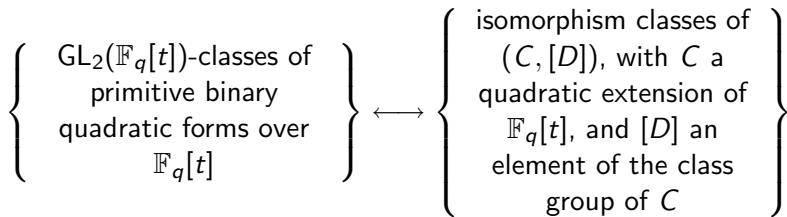
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- $b^2 - 4ac$  is the branch locus of the map from  $C$  to  $\mathbb{A}^1$

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- when  $R = k[x_1, \dots, x_n]$  for a field  $k$  (Quillen–Suslin theorem)

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- taking  $(a, b)$  over  $\mathbb{F}_q[t]$  gives Mumford representation of points on the Jacobian of a hyperelliptic curve



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- over each  $R$  the best method for computation of the composition might differ
- for each  $R$ , the reduction theory to find a unique representative in equivalence classes of forms is a potentially new problem, both theoretically and algorithmically

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Main interesting aspects: reduction theory, efficient implementation

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- compute this class group once, and then compute class groups of many quadratic extensions of  $R$



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- For smooth curves, the class group is the same as the Jacobian

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The story does not stop with cubic extensions (a.k.a. triple covers).

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- $n$ -gonal curves are curves with degree  $n$  covers to the line (here  $\mathbb{A}^1$ )
- as with binary quadratic forms, there is a version of this theorem over any ring (variety, scheme...)

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**Problems:** ( $n \geq 3$ )

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### Problems: ( $n \geq 3$ )

- implement these composition laws explicitly
- understand reduction theory (even with one  $n$ , and one base ring)