Generalizing Vélu's formulas and some applications

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Outline

1. Isogenies
2. Theory
3. Implementation
4. Examples and Applications

Abelian varieties

Definition

An Abelian variety is a complete connected group variety over a base field k.

- (Polarised) abelian varieties = higher dimensional equivalent of elliptic curves.
- If C is a curve of genus g, it's Jacobian is a (principally polarised) abelian variety of dimension g.
- For $C : y^2 = f(x)$ ($\deg f = 2g - 1$) hyperelliptic curve, Mumford coordinates:

$$D = \sum_{i=1}^{k} (P_i - P_\infty) \qquad k \leqslant g, \quad -P_i \neq P_j$$
$$= (u, v) \text{ with } u = \prod (x - x_i), v(x_i) = y_i.$$

Isogenies

Definition

A (separable) isogeny is a finite surjective (separable) morphism between two Abelian varieties.

- Isogenies $\Leftrightarrow$ Finite subgroups.

$$(f : A \to B) \mapsto \operatorname{Ker} f$$
$$(A \to A/H) \leftarrow\!\shortmid H$$

- The kernel of the dual isogeny $\widehat{f}$ is the Cartier dual of the kernel of $f \Rightarrow$ pairings!
- We want isogenies compatible with the polarizations $\Rightarrow$ isotropic kernels.

Cryptographic usage of isogenies

- Transfer the DLP from one Abelian variety to another.
- Point counting algorithms (ℓ-adic or p-adic) $\Rightarrow$ Verify a curve is secure.
- Compute the class field polynomials (CM-method) $\Rightarrow$ Construct a secure curve.
- Compute the modular polynomials $\Rightarrow$ Compute isogenies.
- Determine $\mathrm{End}(A)$ $\Rightarrow$ CRT method for class field polynomials.

Explicit isogeny computation

- Given an isotropic subgroup $K \subset A(\bar{k})$ compute the isogeny $A \mapsto A/K$. (Vélu's formula.)
- Given an abelian variety compute all the isogeneous varieties. (Modular polynomials.)
- Given two isogeneous abelian variety A and B find the isogeny $A \mapsto B$. (''Inverse Vélu's formula'' $\Rightarrow$ SEA algorithm).

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Vélu's formula

Theorem

Let $E : y^2 = f(x)$ be an elliptic curve and $G \subset E(k)$ a finite subgroup. Then E/G is given by $Y^2 = g(X)$ where

$$X(P) = x(P) + \sum_{Q \in G \setminus \{0_E\}} (x(P+Q) - x(Q))$$

$$Y(P) = y(P) + \sum_{Q \in G \setminus \{0_E\}} (y(P+Q) - y(Q)).$$

- Uses the fact that x and y are characterised in $k(E)$ by

$$\begin{aligned} v_{0_E}(x) &= -2 & v_P(x) &\geqslant 0 \quad \text{if } P \neq 0_E \\ v_{0_E}(y) &= -3 & v_P(y) &\geqslant 0 \quad \text{if } P \neq 0_E \\ y^2/x^3(0_E) &= 1 \end{aligned}$$

- No such characterisation in genus $g \geqslant 2$ for Mumford coordinates.

The modular polynomial

Definition

- Modular polynomial $\phi_n(x, y) \in \mathbb{Z}[x, y]$: $\phi_n(x, y) = 0 \Leftrightarrow x = j(E)$ and $y = j(E')$ with E and E' n-isogeneous.
- If $E : y^2 = x^3 + ax + b$ is an elliptic curve, the j-invariant is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

- Roots of $\phi_n(j(E), .) \Leftrightarrow$ elliptic curves n-isogeneous to E.
- In genus 2, modular polynomials use Igusa invariants. The height explodes.

$\Rightarrow$ Genus 2: $(2, 2)$-isogenies [Richelot]. Genus 3: $(2, 2, 2)$-isogenies [Smi09].

$\Rightarrow$ Moduli space given by invariants with more structure.

$\Rightarrow$ Fix the form of the isogeny and look for compatible coordinates.

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Complex abelian varieties and theta functions of level n

- $(\vartheta_i)_{i \in Z(\overline{n})}$: basis of the theta functions of level n. $\quad (Z(\overline{n}) \coloneqq \mathbb{Z}^g / n\mathbb{Z}^g)$
  $\Leftrightarrow A[n] = A_1[n] \oplus A_2[n]$: symplectic decomposition.
- $(\vartheta_i)_{i \in Z(\overline{n})} = \begin{cases} \text{coordinates system} & n \geqslant 3 \\ \text{coordinates on the Kummer variety } A/\pm 1 & n = 2 \end{cases}$
- Theta null point: $\vartheta_i(0)_{i \in Z(\overline{n})}$ = modular invariant.

Example ($k = \mathbb{C}$)

Abelian variety over $\mathbb{C}$: $A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega\mathbb{Z}^g)$; $\Omega \in \mathcal{H}_g(\mathbb{C})$ the Siegel upper half space (Ω symmetric, $\operatorname{Im}\Omega$ positive definite).

$$\vartheta_i \coloneqq \Theta \begin{bmatrix} 0 \\ i/n \end{bmatrix} (z, \Omega/n).$$

The differential addition law ($k = \mathbb{C}$)

$$\Big(\sum_{t\in Z(\bar{2})} \chi(t)\vartheta_{i+t}(x+y)\vartheta_{j+t}(x-y)\Big).\Big(\sum_{t\in Z(\bar{2})} \chi(t)\vartheta_{k+t}(0)\vartheta_{l+t}(0)\Big) =$$
$$\Big(\sum_{t\in Z(\bar{2})} \chi(t)\vartheta_{-i'+t}(y)\vartheta_{j'+t}(y)\Big).\Big(\sum_{t\in Z(\bar{2})} \chi(t)\vartheta_{k'+t}(x)\vartheta_{l'+t}(x)\Big).$$

$$\text{where} \quad \chi \in \hat{Z}(\bar{2}), i, j, k, l \in Z(\bar{n})$$
$$(i', j', k', l') = A(i, j, k, l)$$
$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

The isogeny theorem

Theorem

- *Let* $\ell \wedge n = 1$, *and* $\phi : Z(\overline{n}) \to Z(\overline{\ell n}), x \mapsto \ell.x$ *be the canonical embedding. Let* $K_0 = A[\ell]_2 \subset A[\ell n]_2$.
- *Let* $(\vartheta_i^A)_{i \in Z(\overline{\ell n})}$ *be the theta functions of level* ℓn *on* $A = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$.
- *Let* $(\vartheta_i^B)_{i \in Z(\overline{n})}$ *be the theta functions of level* n *of* $B = A/K_0 = \mathbb{C}^g/(\mathbb{Z}^g + \frac{\Omega}{\ell}\mathbb{Z}^g)$.
- *We have:*

$$(\vartheta_i^B(x))_{i \in Z(\overline{n})} = (\vartheta_{\phi(i)}^A(x))_{i \in Z(\overline{n})}$$

Example

$\pi : (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}) \mapsto (x_0, x_3, x_6, x_9)$ is a 3-isogeny between elliptic curves.

The modular space of theta null points of level n ($\mathrm{car}\, k \nmid n$)

Definition

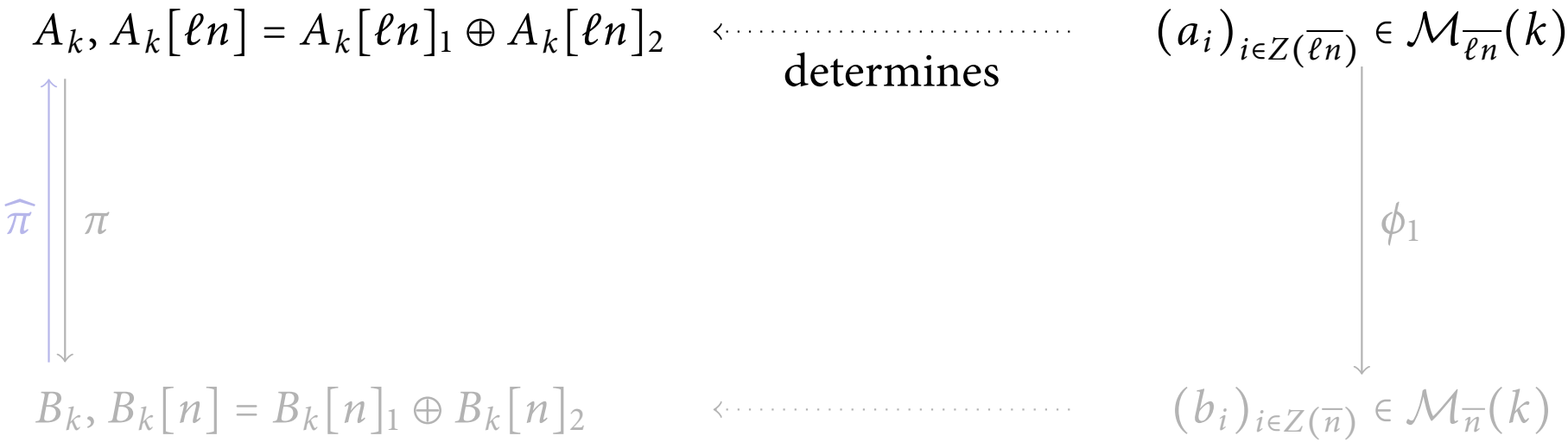
The modular space $\mathcal{M}_{\overline{n}}$ of theta null points is:

$$\sum_{t \in Z(\overline{2})} a_{x+t} a_{y+t} \sum_{t \in Z(\overline{2})} a_{u+t} a_{v+t} = \sum_{t \in Z(\overline{2})} a_{x'+t} a_{y'+t} \sum_{t \in Z(\overline{2})} a_{u'+t} a_{v'+t},$$

with the relations of symmetry $a_x = a_{-x}$.

- Abelian varieties with a n-structure = open locus of $\mathcal{M}_{\overline{n}}$.

Isogenies and modular correspondence [FLR09]



- Every isogeny (with isotropic kernel K) comes from a modular solution.
- We can detect degenerate solutions.

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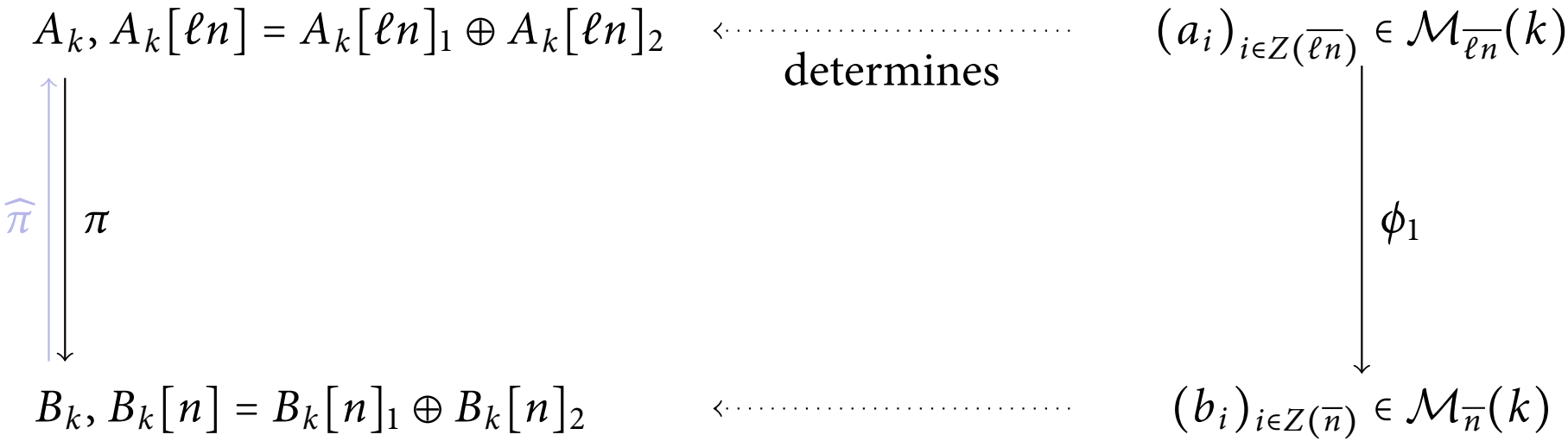

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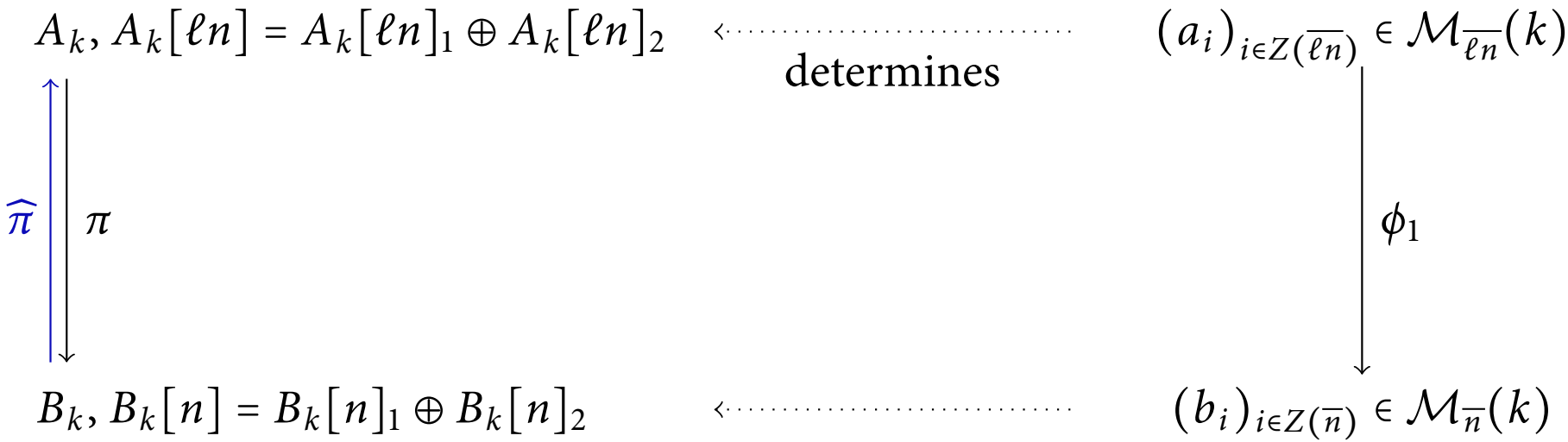

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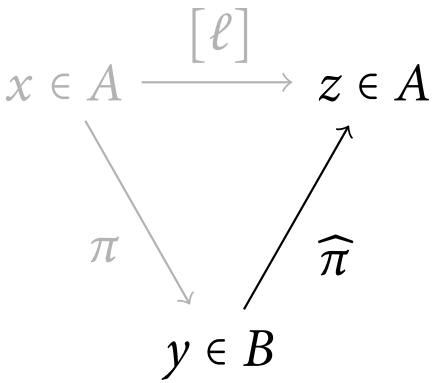
The contragredient isogeny [LR10a]



Let $\pi : A \to B$ be the isogeny associated to $(a_i)_{i \in Z(\overline{\ell n})}$. Let $y \in B$ and $x \in A$ be one of the ℓ^g antecedents. Then

$$\widehat{\pi}(y) = \ell.x$$

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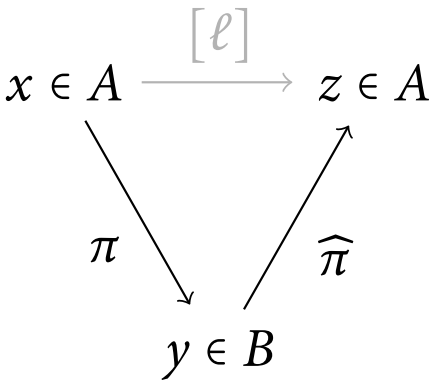

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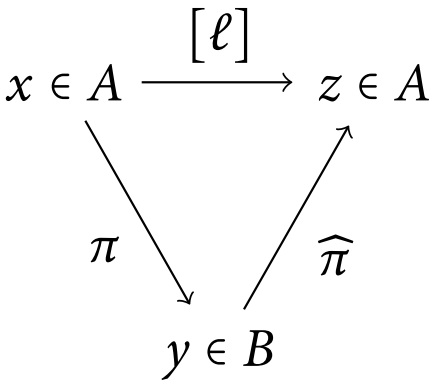

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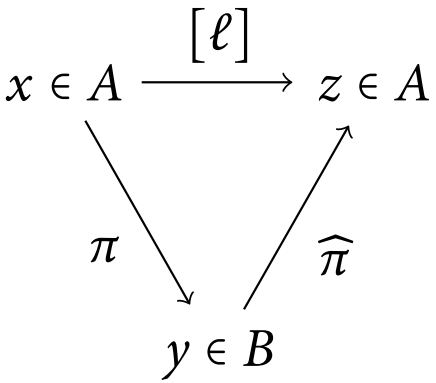

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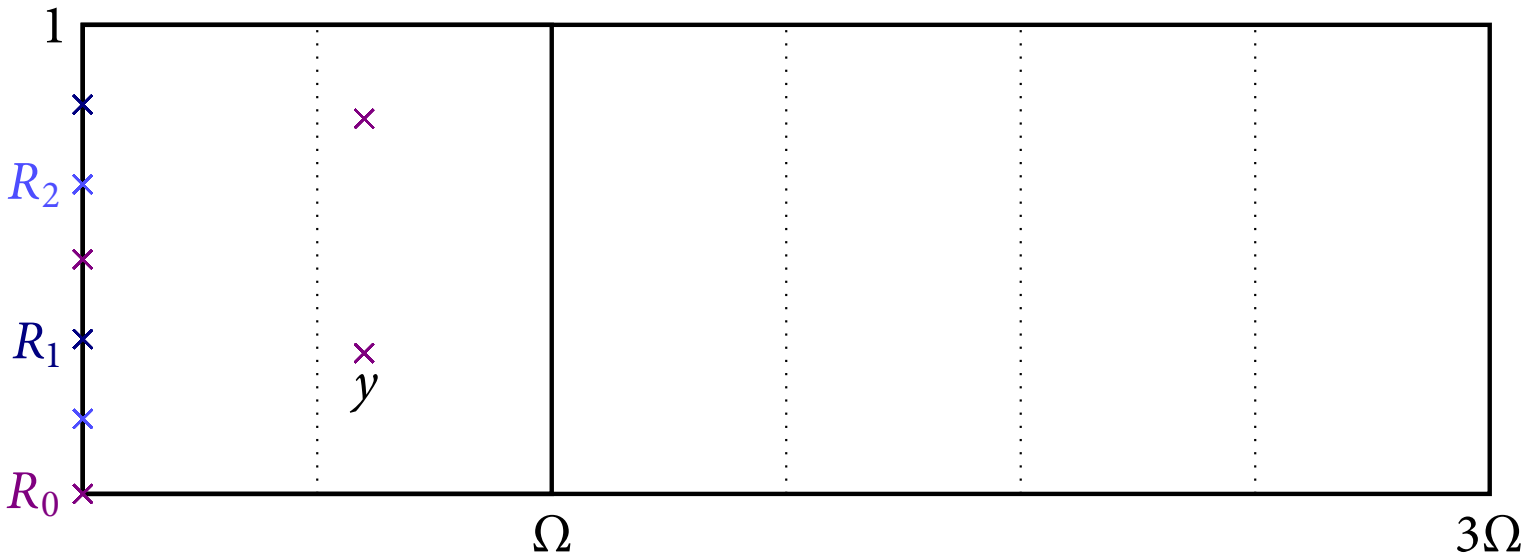

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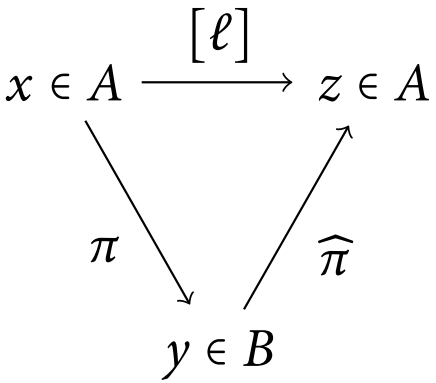

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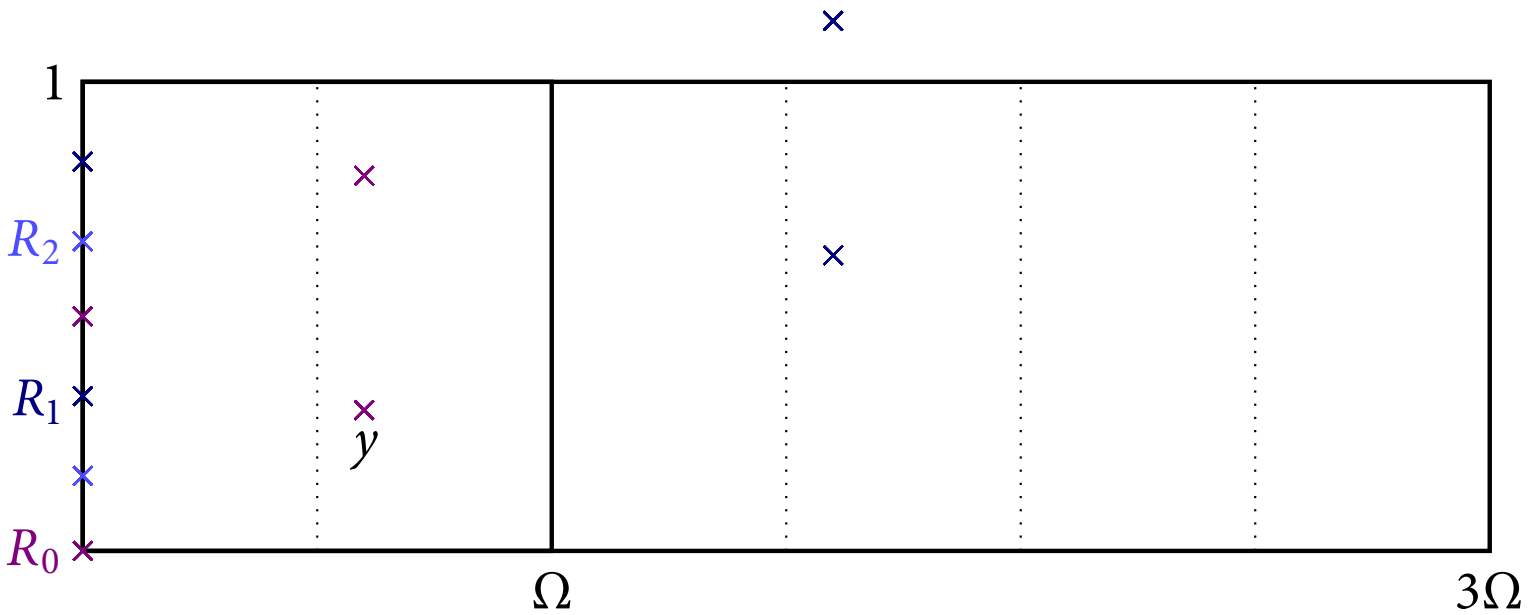

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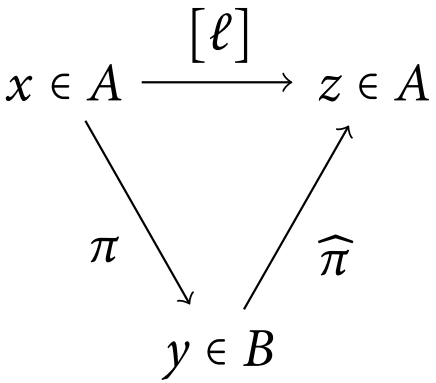

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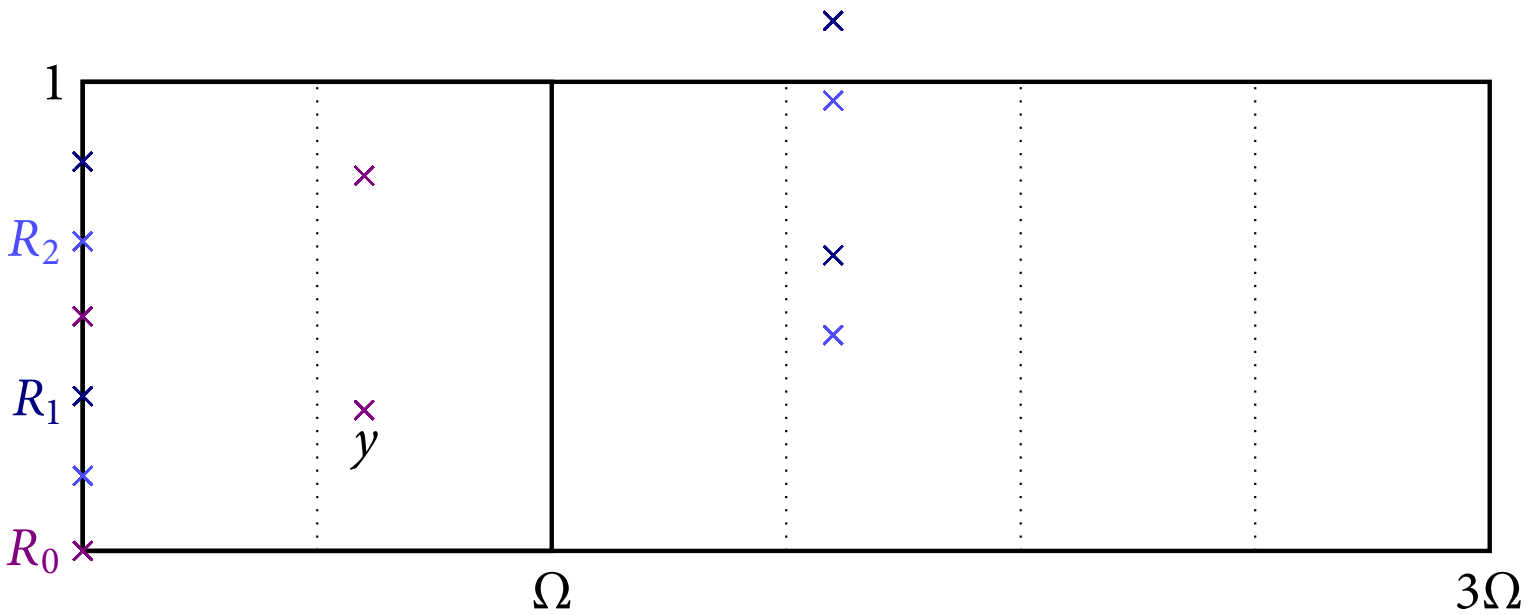

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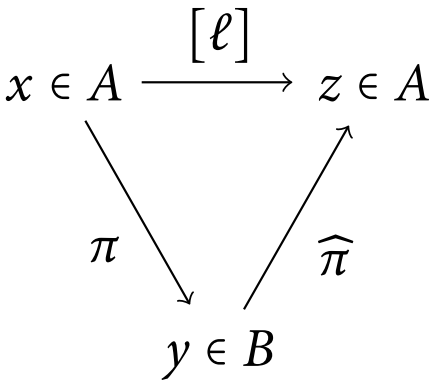

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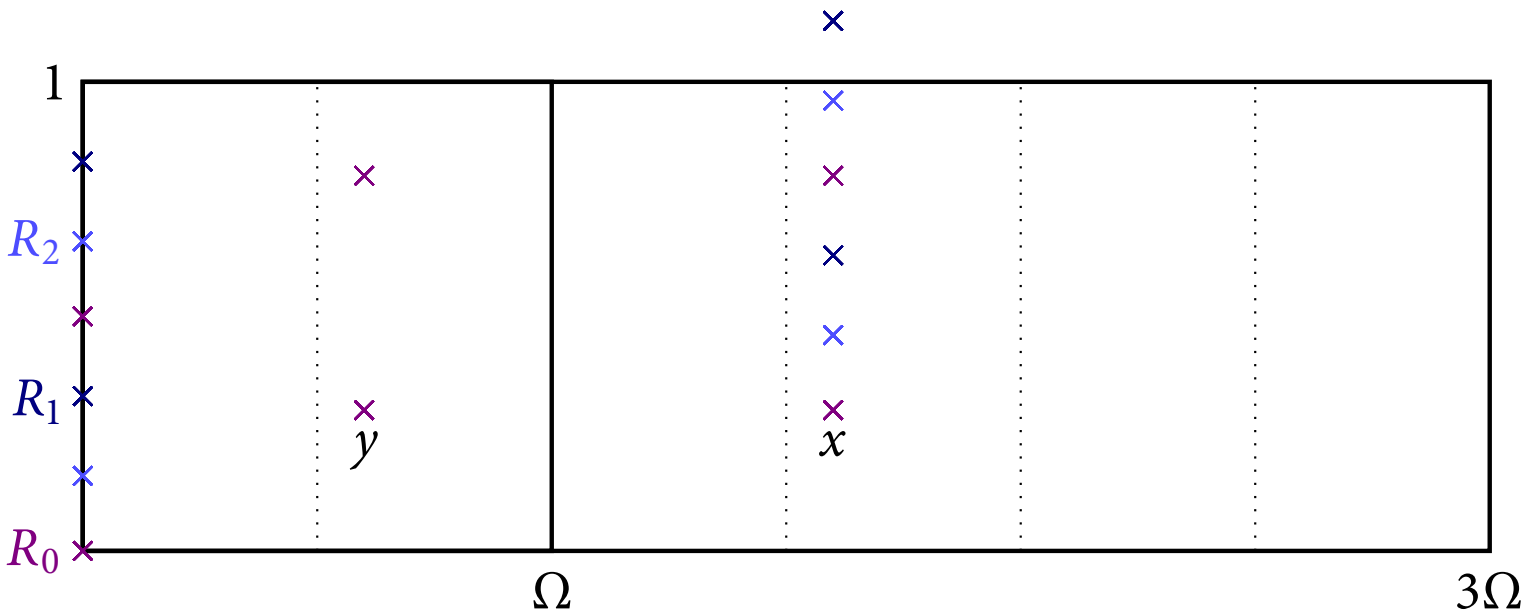

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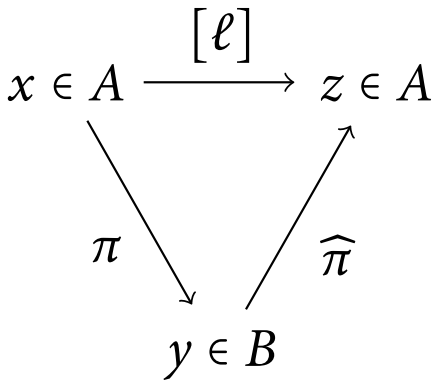

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Explicit isogenies algorithm

- (Compressed) modular point from K: $g(g+1)/2$ ℓ^{th}-roots and $g(g+1)/2 \cdot O(\log(\ell))$ chain additions.

$\Rightarrow$ (Compressed) isogeny: $g \cdot O(\log(\ell))$ chain additions.

Example

- B: elliptic curve $y^2 = x^3 + 23x + 3$ over $k = \mathbb{F}_{31}$

  $\Rightarrow$ Theta null point of level 4: $(3 : 1 : 18 : 1) \in \mathcal{M}_4(\mathbb{F}_{31})$.

- $K = \{(3 : 1 : 18 : 1), (22 : 15 : 4 : 1), (18 : 29 : 23 : 1)\} \Rightarrow$ modular solution:

  $(3, \eta^{14233}, \eta^{2317}, 1, \eta^{1324}, \eta^{5296}, 18, \eta^{5296}, \eta^{1324}, 1, \eta^{2317}, \eta^{14233})$
  $(\eta^3 + \eta + 28 = 0)$.

- $y = (\eta^{19406}, \eta^{19805}, \eta^{10720}, 1)$; $\quad \widehat{\pi}(y)$?

Example

$$R_1 = (\eta^{1324}, \eta^{5296}, \eta^{2317}, \eta^{14233}) \quad y = (\eta^{19406}, \eta^{19805}, \eta^{10720}, 1)$$

$$y \oplus R_1 = \lambda_1(\eta^{2722}, \eta^{28681}, \eta^{26466}, \eta^{2096})$$

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$$y + 3R_1 = \lambda_1^3(\eta^{18374}, \eta^{18773}, \eta^{9688}, \eta^{28758}) = y/\eta^{1032} \quad \text{so } \lambda_1^3 = \eta^{28758}$$

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$$2y + R_1 = \lambda_1^2(\eta^{17786}, \eta^{12000}, \eta^{16630}, \eta^{365})$$

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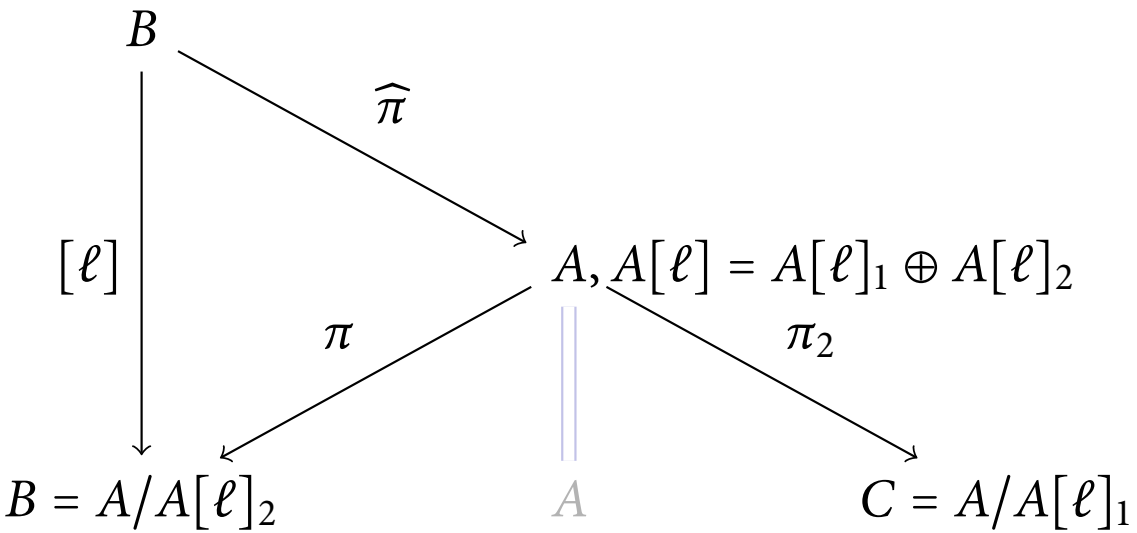
$$\widehat{\pi}(y) = (3, \eta^{21037}, \eta^{15925}, 1, \eta^{8128}, \eta^{18904}, 18, \eta^{12100}, \eta^{14932}, 1, \eta^{9121}, \eta^{27841})$$

Changing level by taking an isogeny



- $\pi_2 \circ \widehat{\pi}$: ℓ^2 isogeny in level n.
- Modular points (corresponding to K) $\Leftrightarrow A[\ell] = A[\ell]_1 \oplus \widehat{\pi}(B[\ell])$
  $\Leftrightarrow \ell^2$-isogenies $B \to C$.

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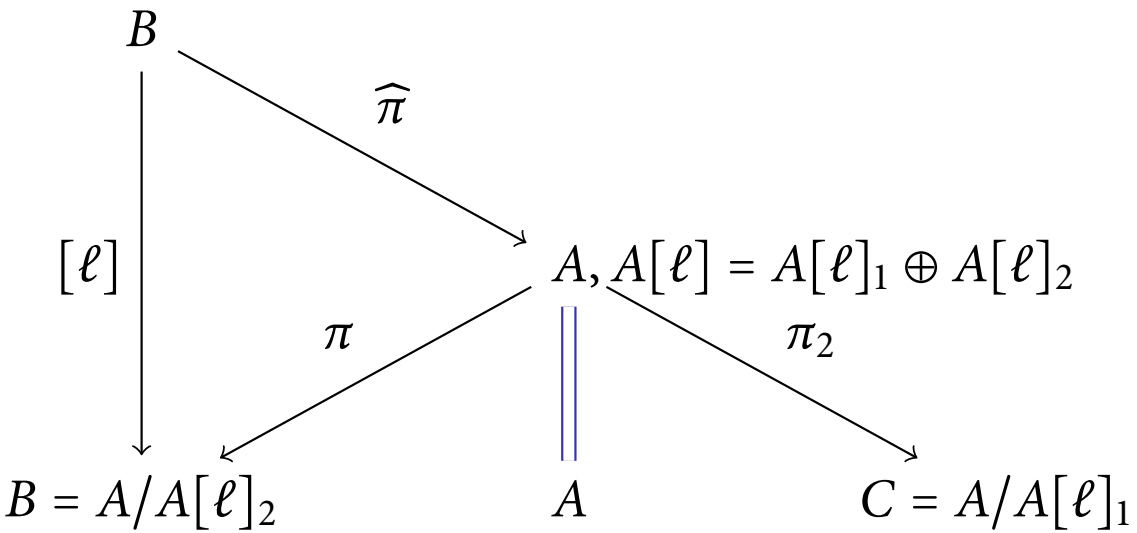

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Changing level without taking isogenies

Theorem (Koizumi-Kempf)

- *Let $\mathcal{L}$ be the space of theta functions of level ℓn and $\mathcal{L}'$ the space of theta functions of level n.*
- *Let $F \in \mathrm{M}_r(\mathbb{Z})$ be such that ${}^tFF = \ell\,\mathrm{Id}$, and $f : A^r \to A^r$ the corresponding isogeny.*

We have $\mathcal{L} = f^*\mathcal{L}'$ and the isogeny f is given by

$$f^*(\vartheta_{i_1}^{\mathcal{L}'} \star \ldots \star \vartheta_{i_r}^{\mathcal{L}'}) = \lambda \sum_{\substack{(j_1,\ldots,j_r)\in K_1(\mathcal{L}')\times\ldots\times K_1(\mathcal{L}') \\ f(j_1,\ldots,j_r)=(i_1,\ldots,i_r)}} \vartheta_{j_1}^{\mathcal{L}} \star \ldots \star \vartheta_{j_r}^{\mathcal{L}}$$

- $F = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ gives the Riemann relations. (For general ℓ, use the quaternions.)

$\Rightarrow$ Go up and down in level without taking isogenies [Cosset+R].

A complete generalisation of Vélu's algorithm [Cosset+R]

- Compute the isogeny $B \to A$ while staying in level n.
- No need of ℓ-roots. Need only $O(\#K)$ differential additions in B + $O(\ell^g)$ or $O(\ell^{2g})$ multiplications $\Rightarrow$ fast.
- The formulas are rational if the kernel K is rational.

- Blocking part: compute $K \Rightarrow$ compute all the ℓ-torsion on B.
  $g = 2$: ℓ-torsion, $\widetilde{O}(\ell^6)$ vs $O(\ell^2)$ or $O(\ell^4)$ for the isogeny.

$\Rightarrow$ Work in level 2.

$\Rightarrow$ Convert back and forth to Mumford coordinates:

$$\begin{array}{ccc} B & \xrightarrow{\widehat{\pi}} & A \\ \| & & \| \\ \mathrm{Jac}(C_1) & \cdots\cdots\cdots\rightarrow & \mathrm{Jac}(C_2) \end{array}$$

Avisogenies

- Avisogenies: Magma code written by Bisson, Cosset and R.
- Released under LGPL 2+.
- Implement isogeny computation (and applications thereof) for abelian varieties using theta functions.
- Current alpha release: isogenies in genus 2.

Implementation

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4. For each kernel K, convert its basis from Mumford to theta coordinates of level 2. (Rosenhain then Thomae).
5. Compute the other points in K in theta coordinates using differential additions.
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8. Distinguish between the isogeneous curve and its twist.

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Computing the right extension

- $J = \mathrm{Jac}(H)$ abelian variety of dimension 2. $\chi(X)$ the corresponding zeta function.
- Degree of a point of ℓ-torsion | the order of X in $\mathbb{F}_\ell[X]/\chi(X)$.
- If K rational, $K(\overline{k}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, the degree of a point in K | the LCM of orders of X in $\mathbb{F}_\ell[X]/P(X)$ for $P \mid \chi$ of degree two.
- Since we are looking to K maximal isotropic, $J[\ell] \simeq K \oplus K'$ and we know that $P \mid \chi$ is such that $\chi(X) \equiv P(X)P(\overline{X}) \mod \ell$ where $\overline{X} = q/X$ represents the Verschiebung.

Remark

The degree n is $\leqslant \ell^2 - 1$. If ℓ is totally split in $\mathbb{Z}[\pi, \overline{\pi}]$ then $n \mid \ell - 1$.

Computing the ℓ-torsion

- We want to compute $J(\mathbb{F}_{q^n})[\ell]$.
- From the zeta function $\chi(X)$ we can compute random points in $J(\mathbb{F}_{q^n})[\ell^\infty]$ uniformly.
- If P is in $J(\mathbb{F}_{q^n})[\ell^\infty]$, $\ell^m P \in J(\mathbb{F}_{q^n})[\ell]$ for a suitable m. This does not give uniform points of ℓ-torsion but we can correct the points obtained.

Example

- Suppose $J(\mathbb{F}_{q^n})[\ell^\infty] =< P_1, P_2 >$ with P_1 of order ℓ^2 and P_2 of order ℓ.
- First random point $Q_1 = P_1 \Rightarrow$ we recover the point of ℓ-torsion: $\ell.P_1$.
- Second random point $Q_2 = \alpha P_1 + \beta P_2$. If $\alpha \neq 0$ we recover the point of ℓ-torsion $\alpha \ell P_1$ which is not a new generator.
- We correct the original point: $Q_2' = Q_2 - \alpha Q_1 = \beta P_2$.

Weil pairing

- Used to decompose a point $P \in J[\ell]$ in term of a basis of the ℓ-torsion (and to construct a symplectic basis).
- The magma implementation is **extremely** slow in genus 2 for non degenerate divisors.
- But since we convert the points in theta coordinates we can use the pairing in theta coordinates [LR10b].

Timings for isogenies computations ($\ell = 7$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = t^254*x^6 + t^223*x^5 +
  t^255*x^4 + t^318*x^3 + t^668*x^2 + t^543*x + t^538 over GF(3^6)
> time RationallyIsogenousCurvesG2(J,7);
** Computing  7 -rationnal isotropic subgroups
  -- Computing the 7 -torsion over extension of deg 4
  !! Basis: 2 points in Finite field of size 3^24
  -- Listing subgroups
  1 subgroups over Finite field of size 3^24
  -- Convert the subgroups to theta coordinates
  Time: 0.060
Computing the 1 7 -isogenies
  ** Precomputations for l= 7 Time: 0.180
  ** Computing the 7 -isogeny
    Computing the l-torsion Time: 0.030
    Changing level Time: 0.210
  Time: 0.430
Time: 0.490
[ <[ t^620, t^691, t^477 ], Jacobian of Hyperelliptic Curve defined by
y^2 = t^615*x^6 + t^224*x^5 + t^37*x^4 + t^303*x^3 + t^715*x^2 + t^128*x
```

Timings for isogenies computations ($\ell = 5$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = 39*x^6 + 4*x^5 + 82*x^4
  + 10*x^3 + 31*x^2 + 39*x + 2 over GF(83)
> time RationallyIsogenousCurvesG2(J,5);
** Computing  5 -rationnal isotropic subgroups
  -- Computing the 5 -torsion over extension of deg 24
  Time: 0.940
  !! Basis: 4 points in Finite field of size 83^24
  -- Listing subgroups
  Time: 1.170
  6 subgroups over Finite field of size 83^24
  -- Convert the subgroups to theta coordinates
  Time: 0.360
Time: 2.630
Computing the 6 5 -isogenies
Time: 0.820
Time: 3.460
 [ <[ 36, 69, 38 ], Jacobian of Hyperelliptic Curve defined by
 y^2 = 27*x^6 + 63*x^5 + 5*x^4 + 24*x^3 + 34*x^2 + 6*x + 76 over GF(83)>,
   ...]
```

Timings for isogeny graphs ($\ell = 3$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = 41*x^6 + 131*x^5 +
  55*x^4 + 57*x^3 + 233*x^2 + 225*x + 51 over GF(271)
time isograph,jacobians:=IsoGraphG2(J,{3}: save_mem:=-1);
Computed 540 isogenies and found 135 curves.
Time: 14.410
```

- Core 2 with 4BG of RAM.
- Computing kernels: $\approx 5s$.
- Computing isogenies: $\approx 7s$ (Torsion: $\approx 2s$, Changing level: $\approx 3.5s$.)

Going further ($\ell = 53$)

```
Jacobian of Hyperelliptic Curve defined by y^2 = 97*x^6 + 77*x^5 +
  62*x^4 + 14*x^3 + 33*x^2 + 18*x + 40 over GF(113)
> time RationallyIsogenousCurvesG2(J,53);
** Computing  53 -rationnal isotropic subgroups
  -- Computing the 53 -torsion over extension of deg 52 Time: 8.610
  !! Basis: 3 points in Finite field of size 113^52
  -- Listing subgroups Time: 1.210
  2 subgroups over Finite field of size 113^52
  -- Convert the subgroups to theta coordinates Time: 0.100
  Time: 9.980
Computing the 2 53 -isogenies
  ** Precomputations for l= 53 Time: 0.240
  ** Computing the 53 -isogeny
    Computing the l-torsion Time: 7.570
    Changing level Time: 1.170
  Time: 8.840
  ** Computing the 53 -isogeny
  Time: 8.850
Time: 27.950
```

Going further ($\ell = 19$)

```
  Jacobian of Hyperelliptic Curve defined by y^2 = 194*x^6 + 554*x^5 +
  606*x^4 + 523*x^3 + 642*x^2 + 566*x + 112 over GF(859)
  > time RationallyIsogenousCurvesG2(J,19);
  ** Computing  19 -rationnal isotropic subgroups (extension degree 18)
  Time: 0.760
Computing the 2 19 -isogenies
  ** Precomputations for l= 19 Time: 11.160
  ** Computing the 19 -isogeny
    Computing the l-torsion Time: 0.250
    Changing level Time: 18.590
  Time: 18.850
  ** Computing the 19 -isogeny
    Computing the l-torsion Time: 0.250
    Changing level Time: 18.640
  Time: 18.900
Time: 51.060
[ <[ 341, 740, 389 ], Jacobian of Hyperelliptic Curve defined by y^2 = 72
    680*x^5 + 538*x^4 + 613*x^3 + 557*x^2 + 856*x + 628 over GF(859)>,
  ... ]
```

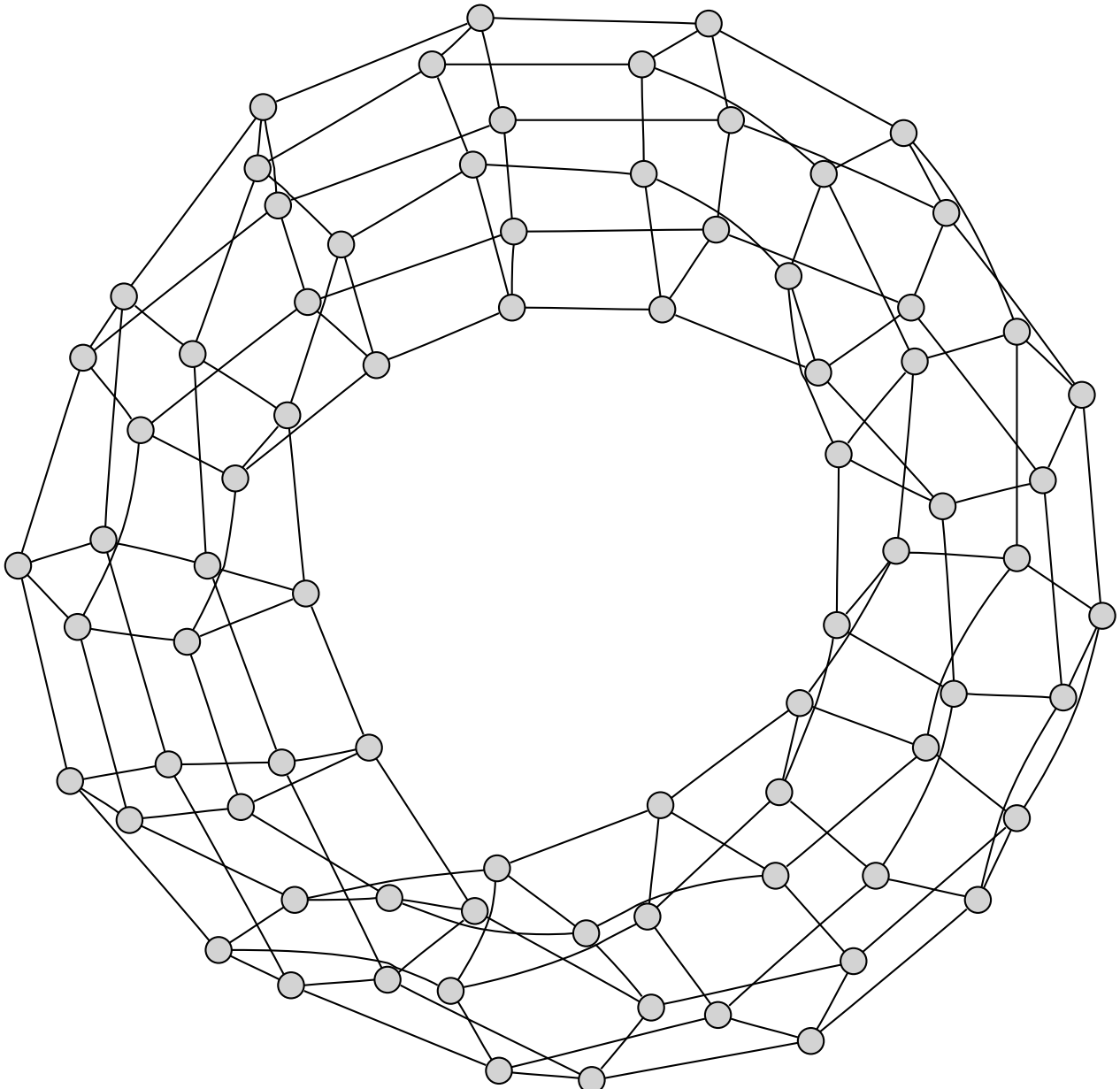
A record isogeny computation! ($\ell = 1321$)

- J Jacobian of $y^2 = x^5 + 41691x^4 + 24583x^3 + 2509x^2 + 15574x$ over $\mathbb{F}_{42179}$.
- $\#J = 2^{10}1321^2$.

```
> time RationallyIsogenousCurvesG2(J,1321:ext_degree:=1);
** Computing  1321 -rationnal isotropic subgroups
Time: 0.350
Computing the 1 1321 -isogenies
  ** Precomputations for l= 1321
  Time: 1276.950
  ** Computing the 1321 -isogeny
    Computing the l-torsion
    Time: 1200.270
    Changing level
    Time: 1398.780
  Time: 5727.250
Time: 7004.240
Time: 7332.650
[ <[ 9448, 15263, 31602 ], Jacobian of Hyperelliptic Curve defined by
  y^2 = 33266*x^6 + 20155*x^5 + 31203*x^4 + 9732*x^3 +
  4204*x^2 + 18026*x + 29732 over GF(42179)> ]
```

- Core 2 with 32GB of RAM

Isogeny graphs: $\ell = q_1 q_2 = Q_1 \overline{Q}_1 Q_2 \overline{Q}_2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



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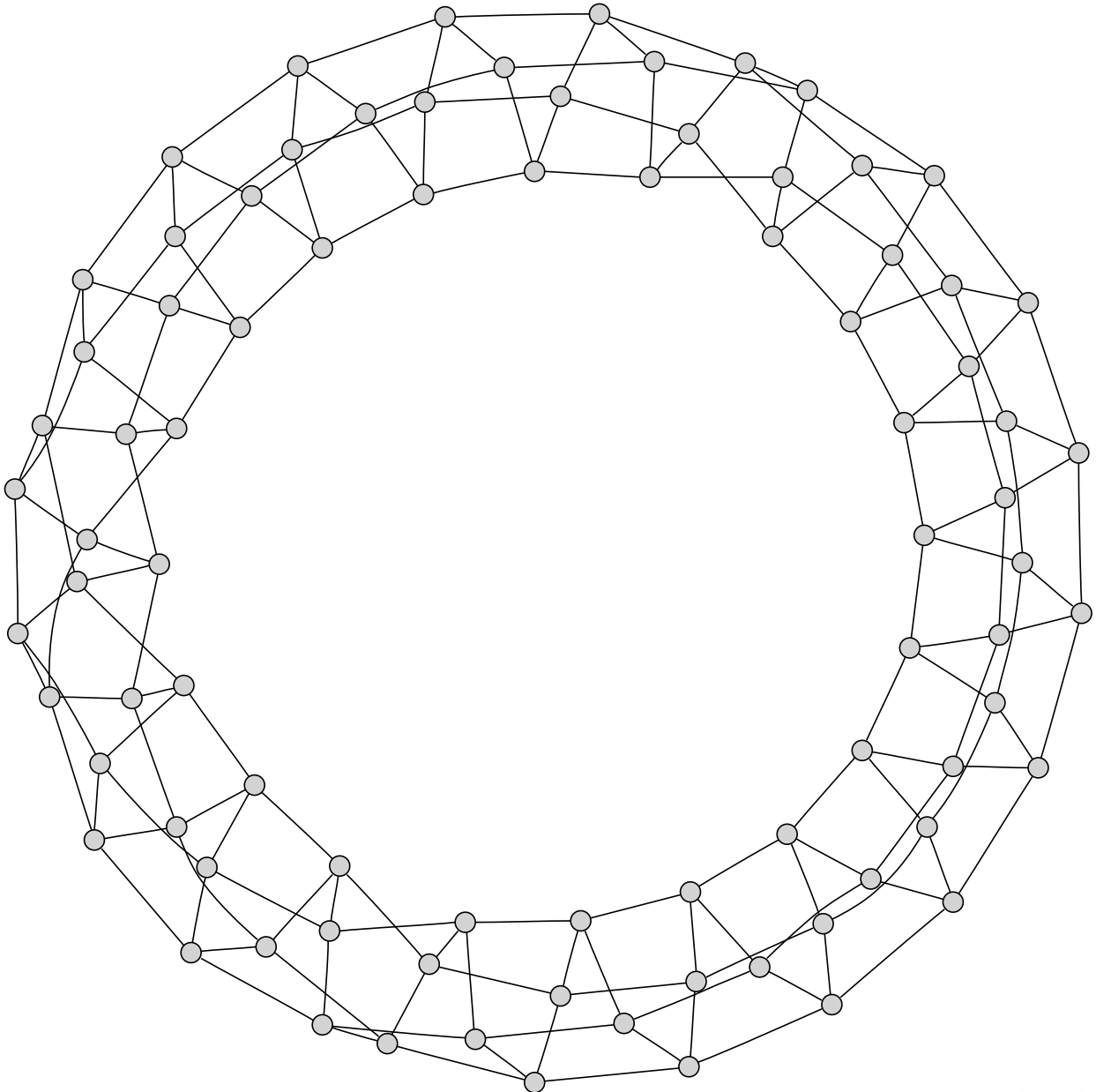
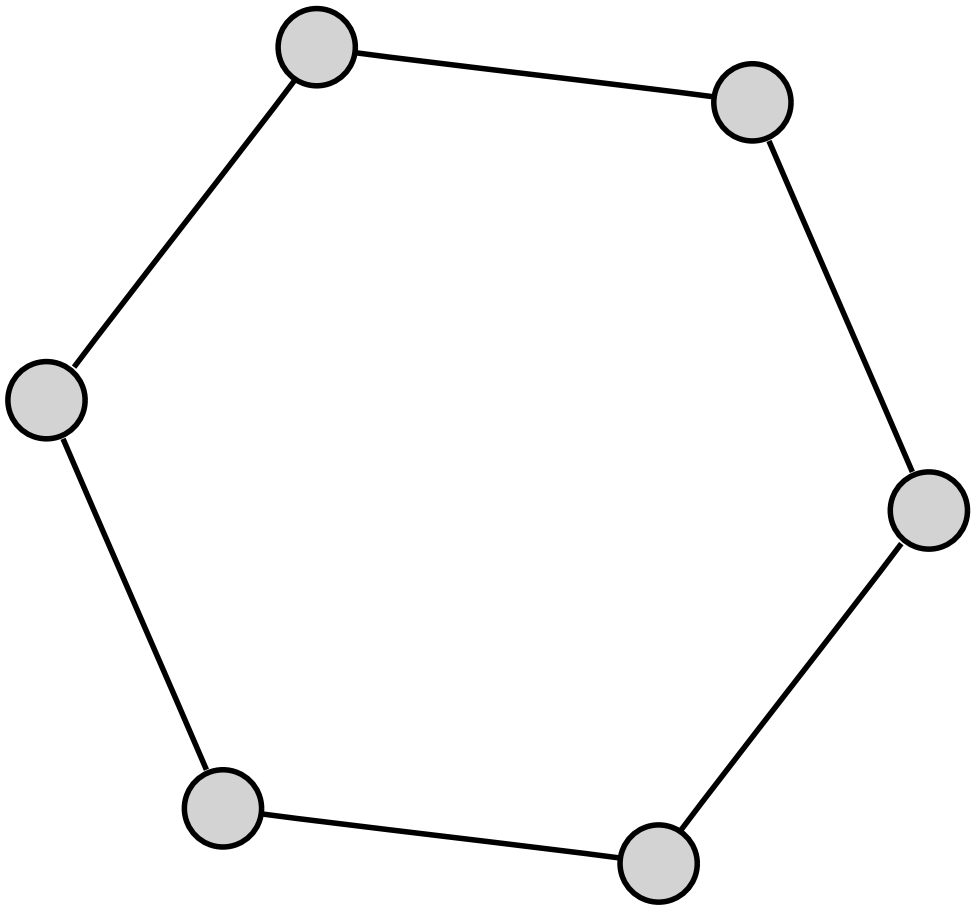


Isogeny graphs: $\ell = q = Q\overline{Q}$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



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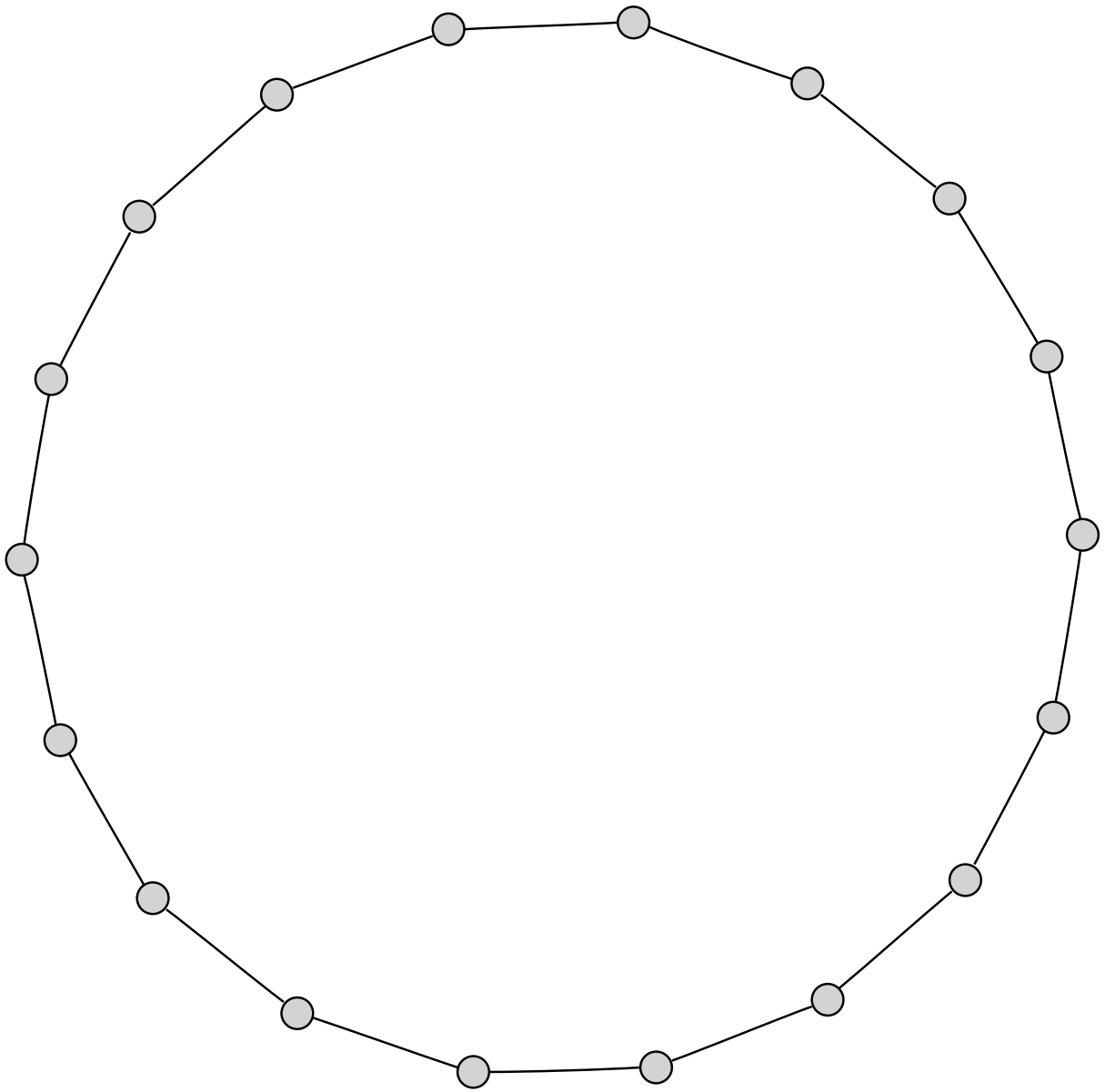
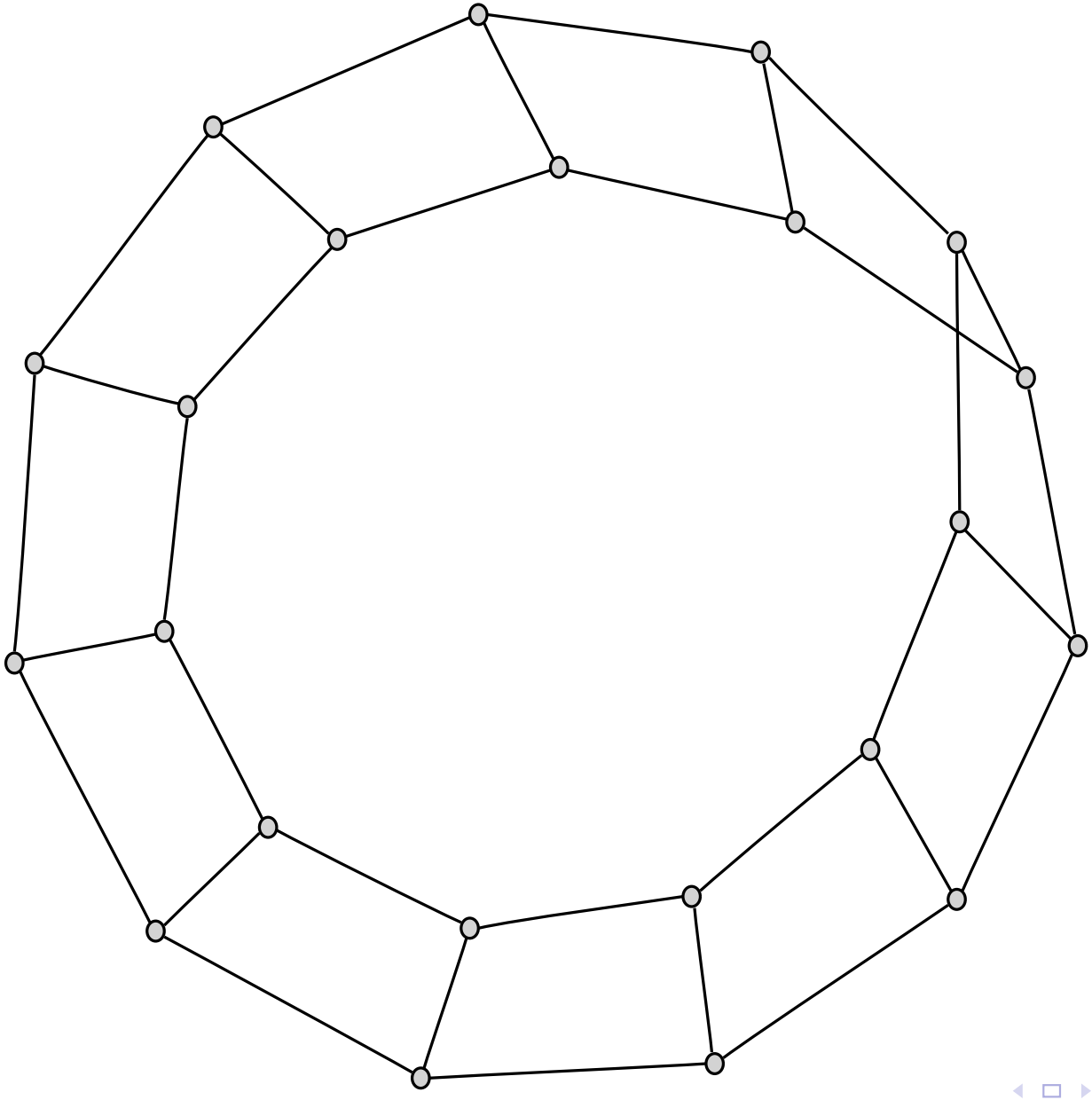
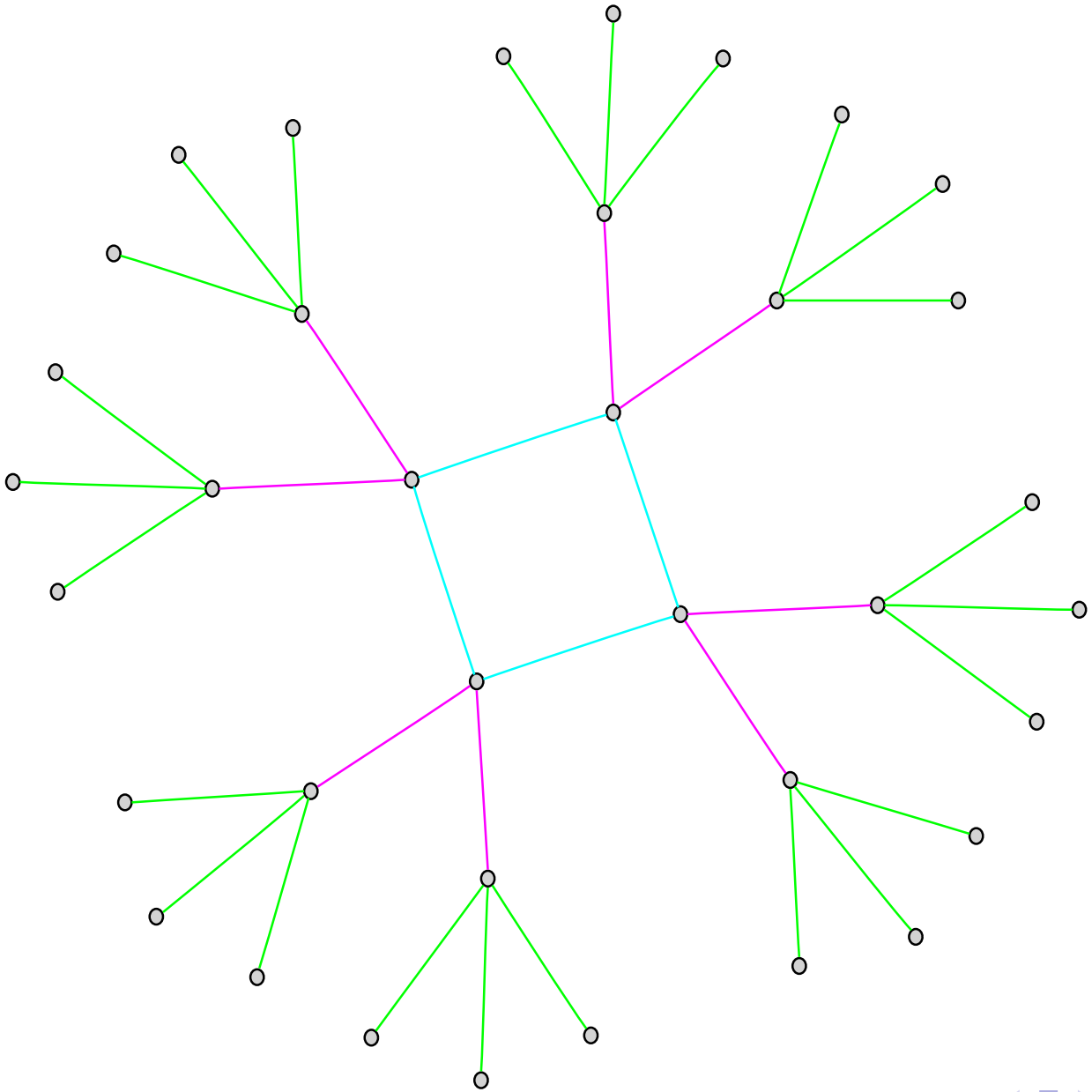


Isogeny graphs: $\ell = q^2 = Q^2\overline{Q}^2$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$



Isogeny graphs: $\ell = q^2 = Q^4$ $(\mathbb{Q} \mapsto K_0 \mapsto K)$

Non maximal isogeny graphs ($\ell = q = Q\overline{Q}$)



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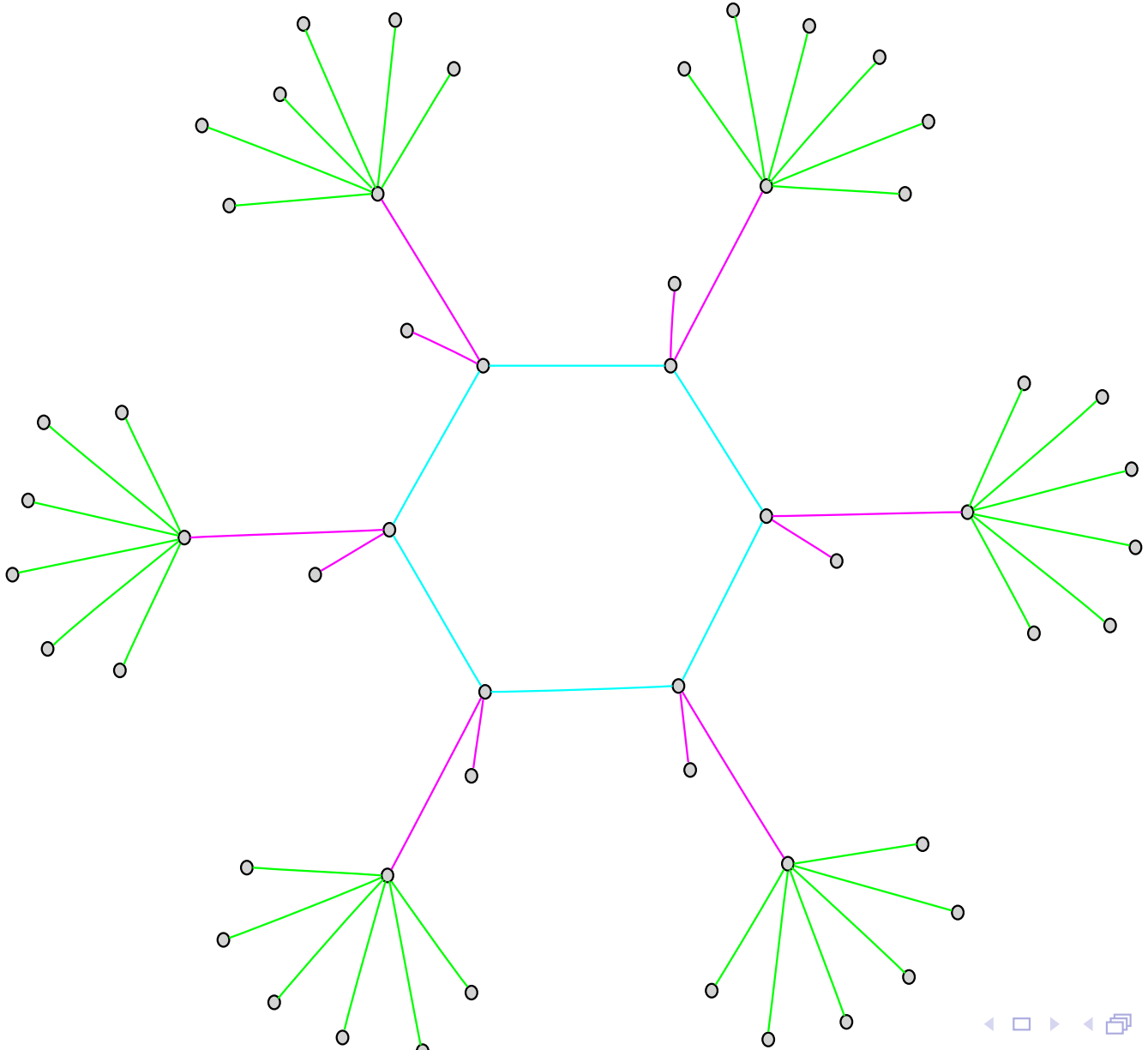

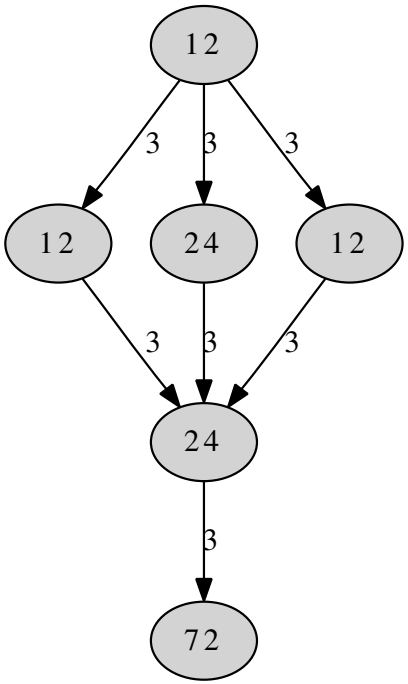

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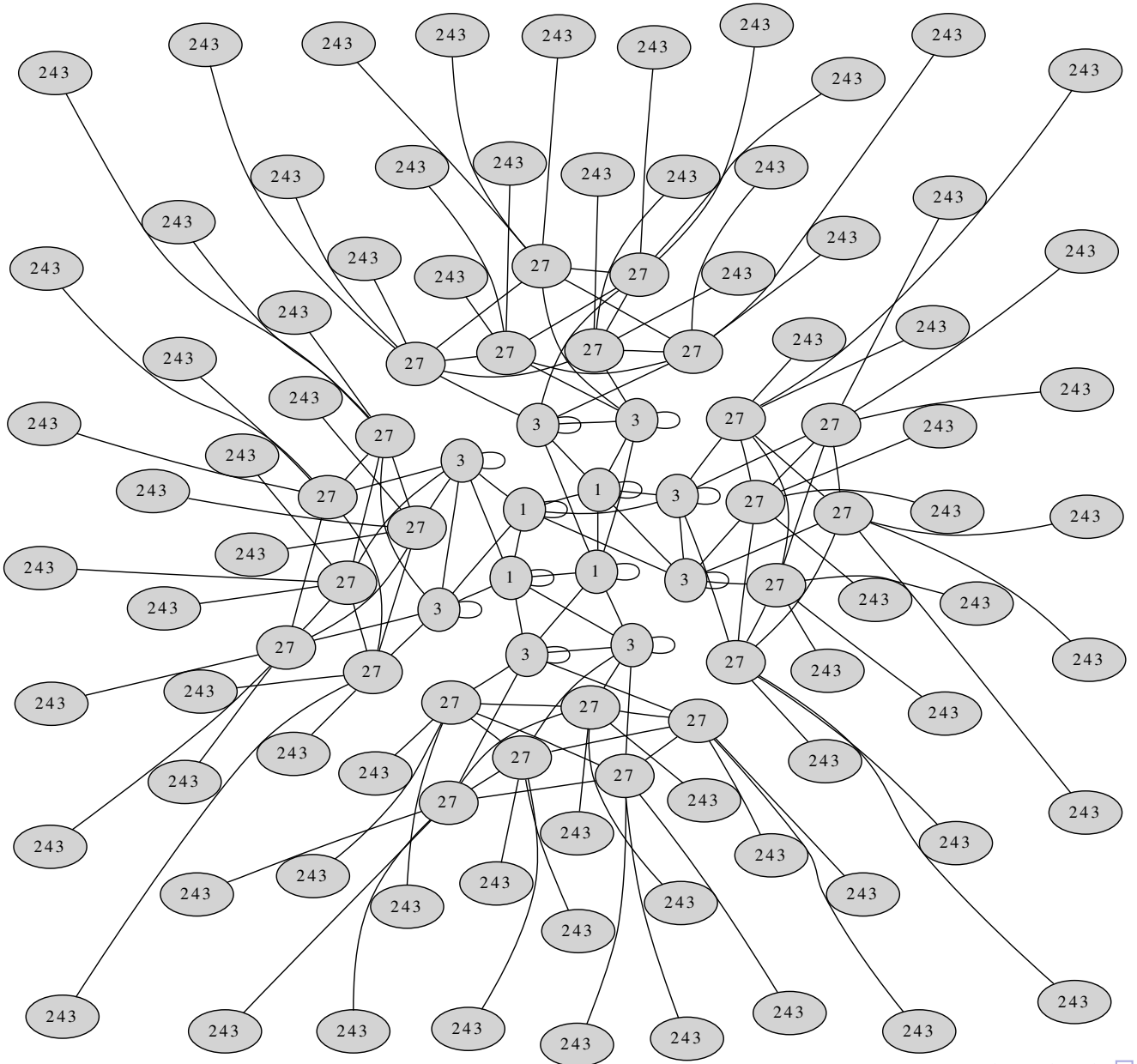

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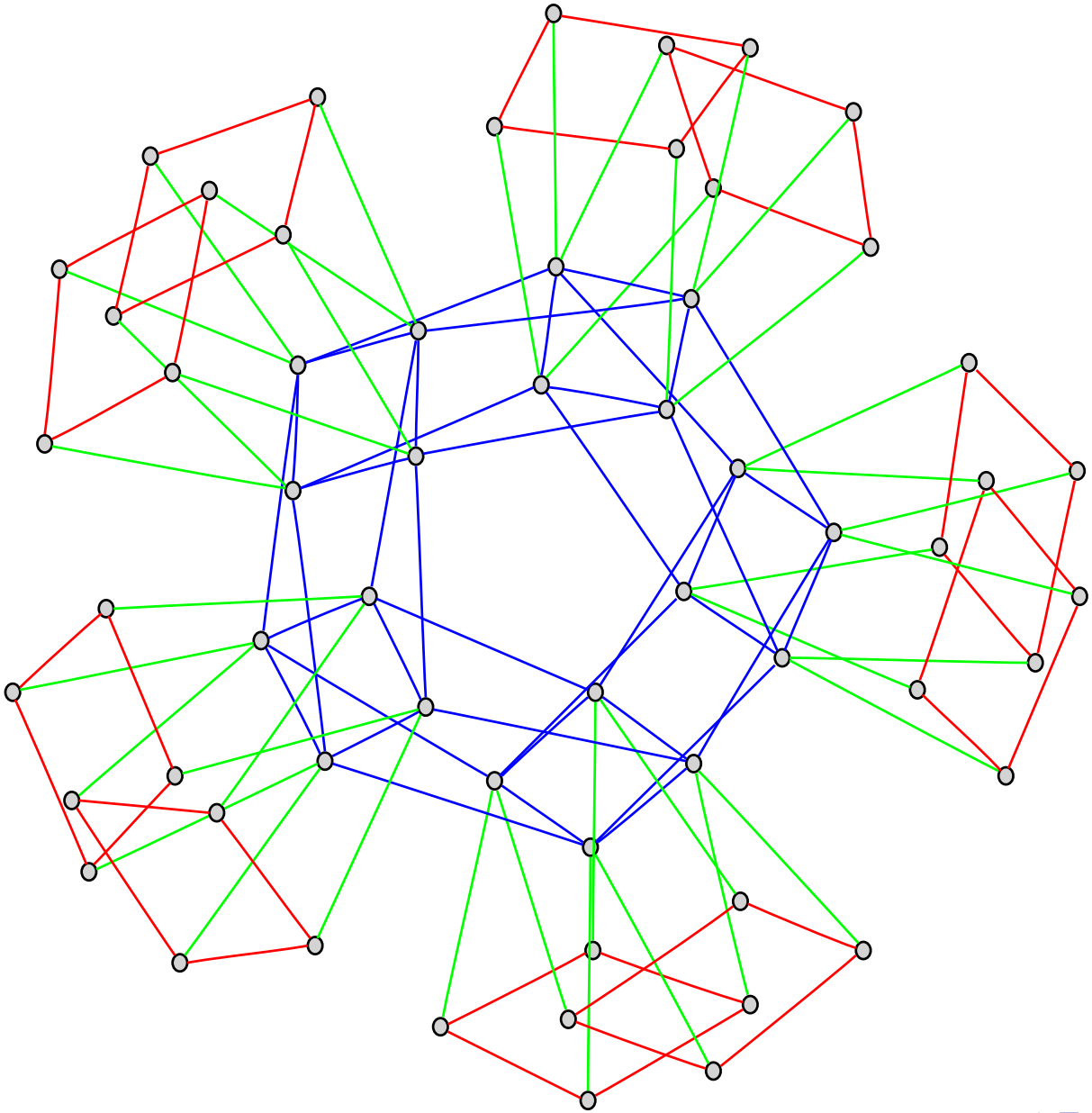

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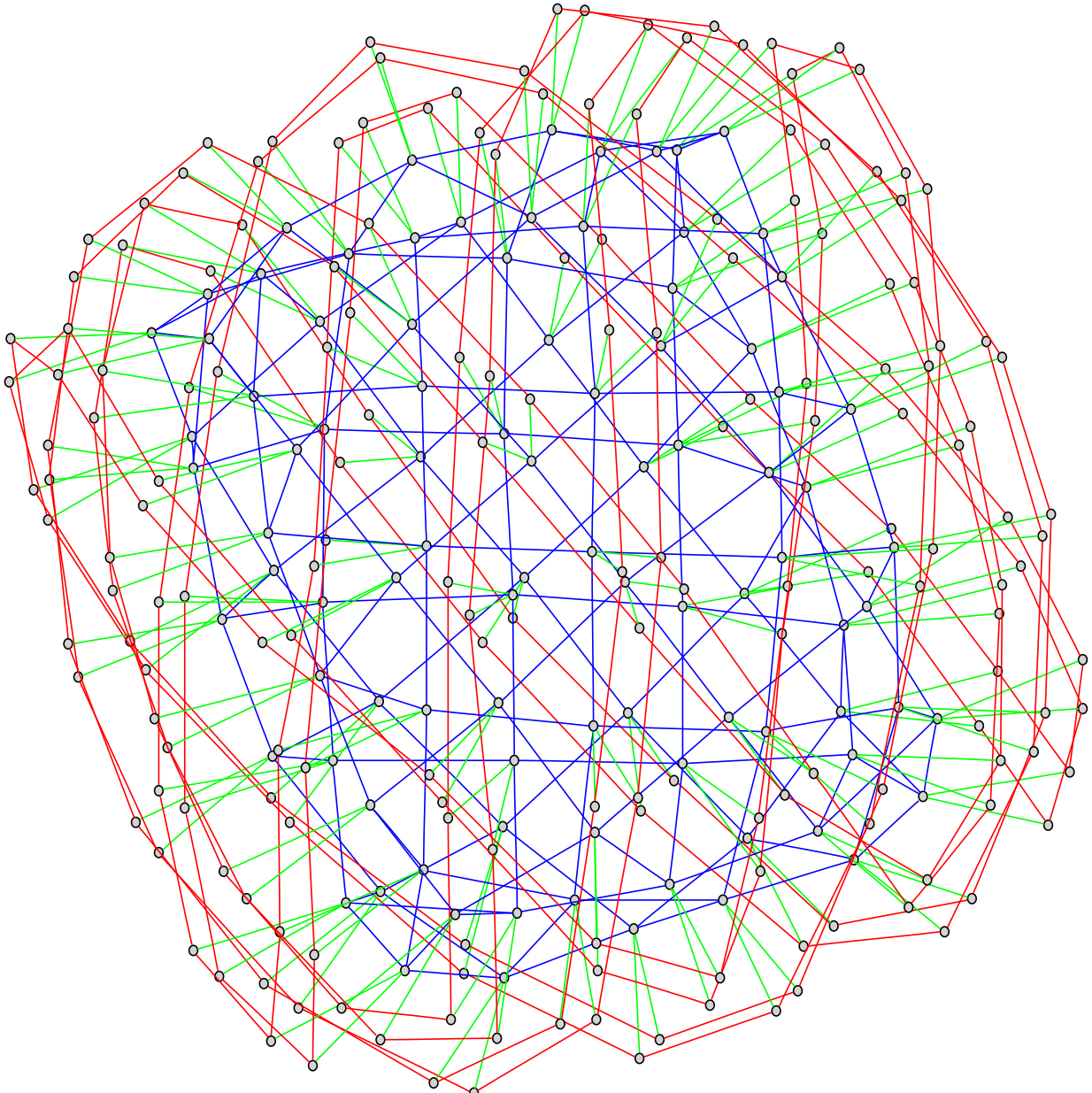

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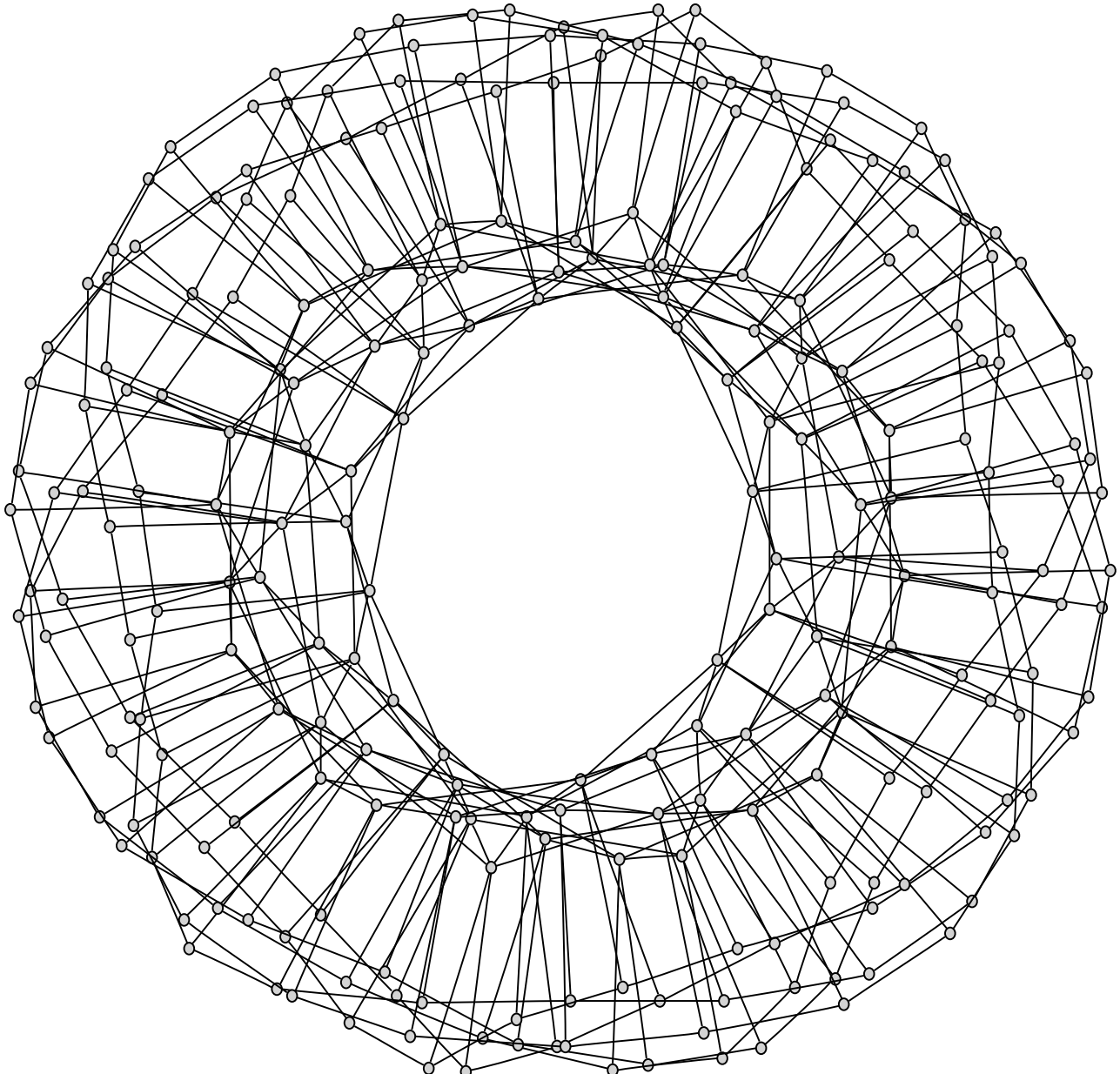

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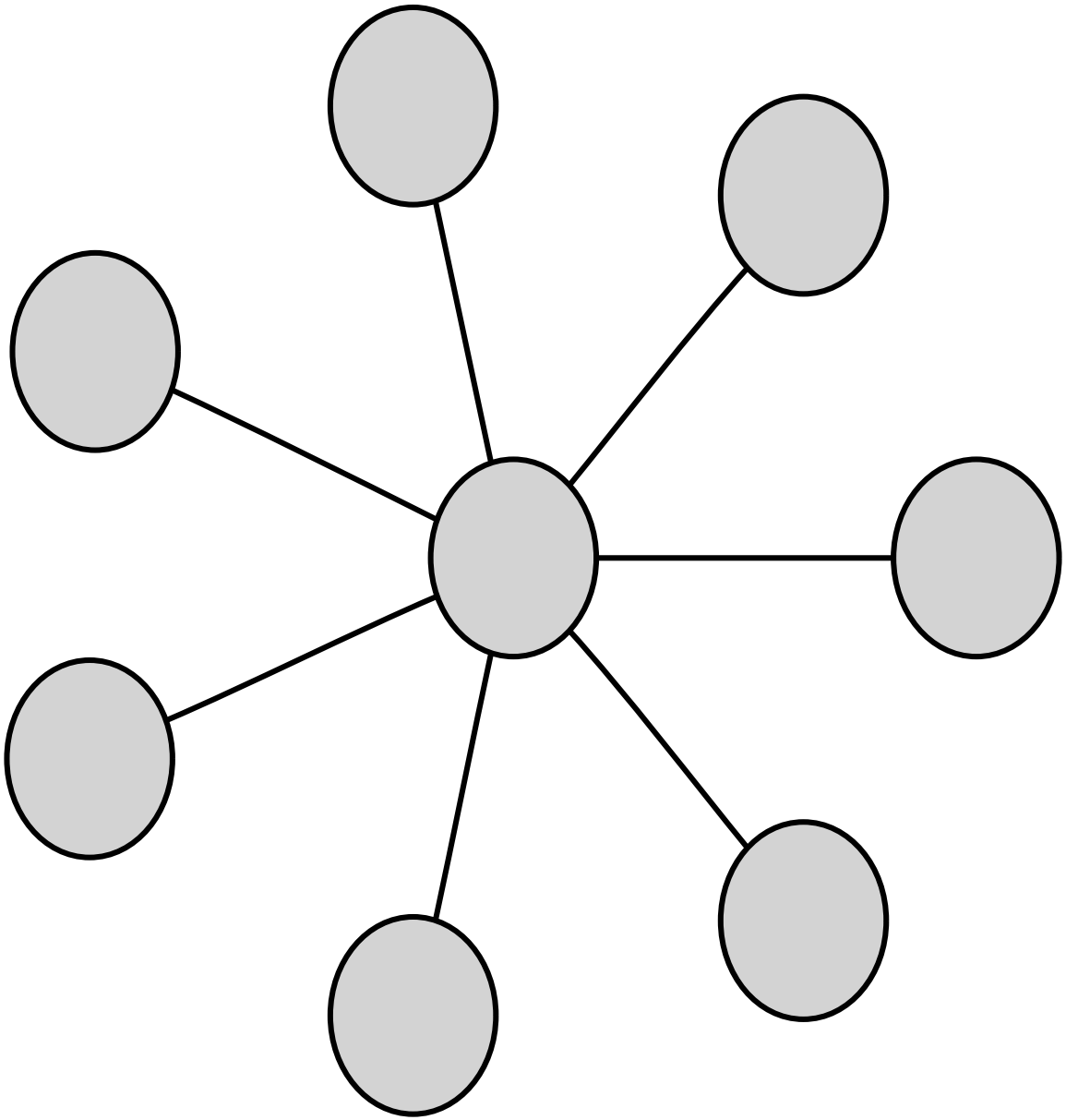

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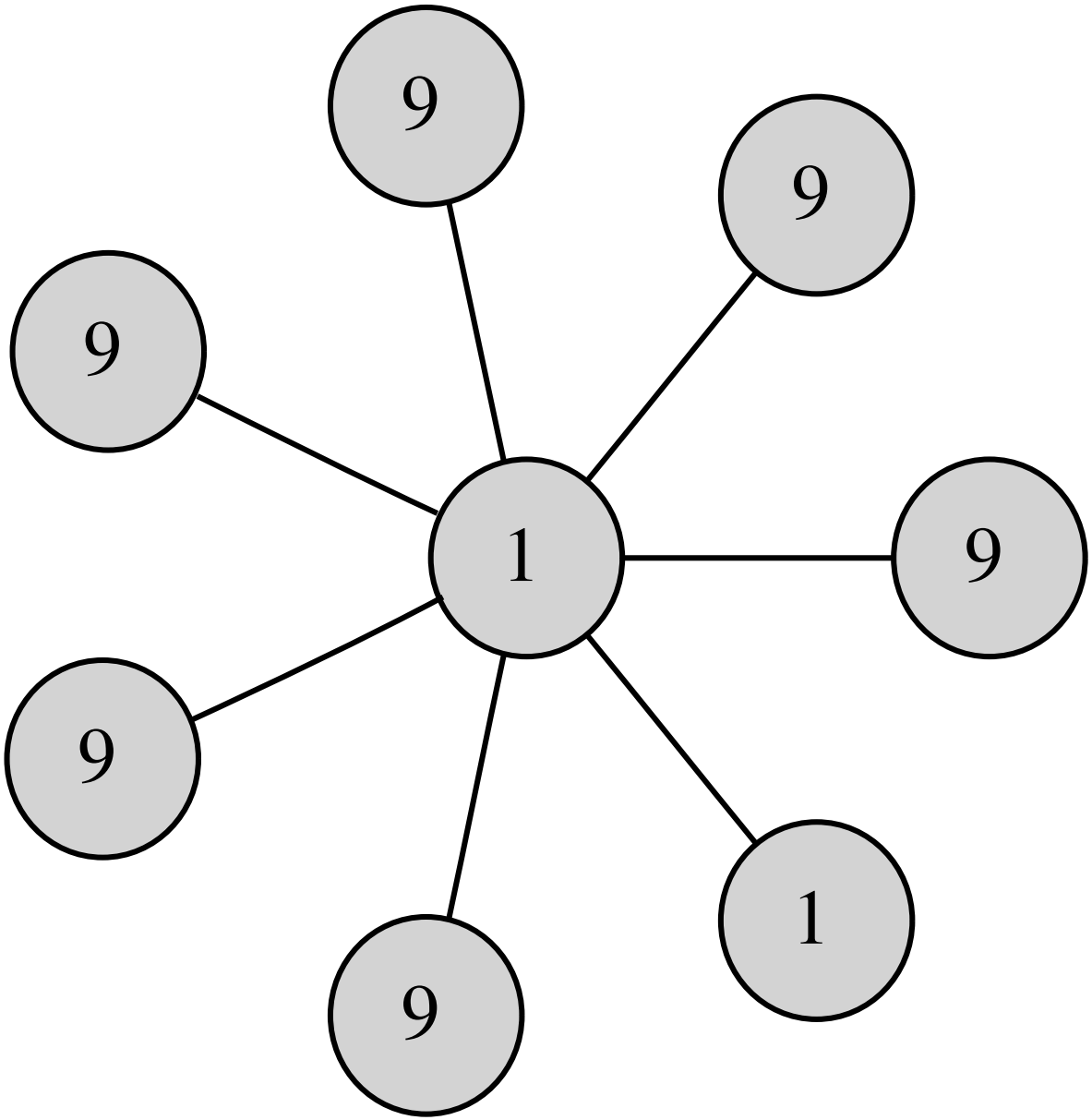


Applications of isogenies to higher genus

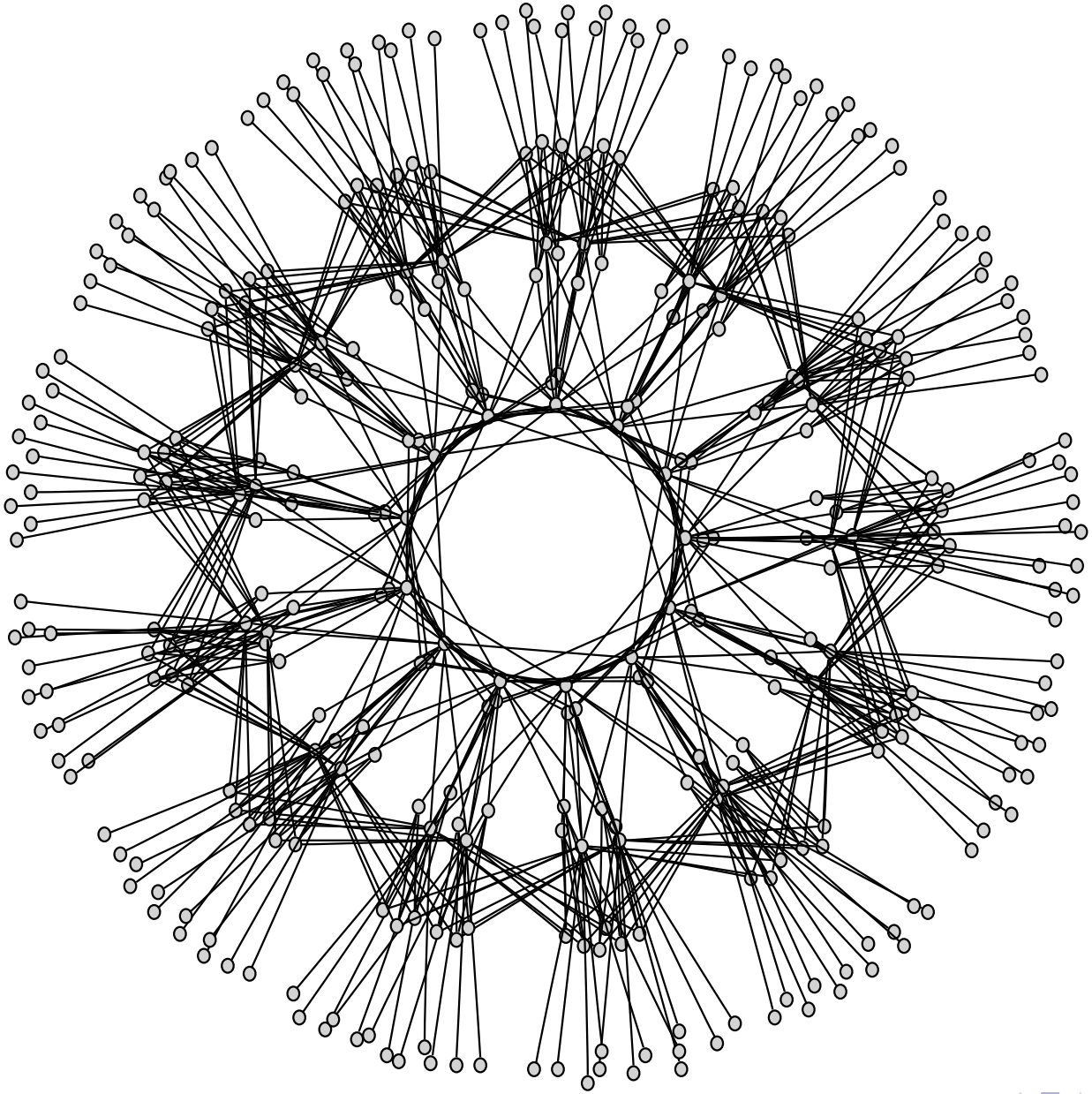
- Computing endomorphism ring. Generalize [BS09] to higher genus, work by Bisson.
- Class polynomials in genus 2 using the CRT. If K is a CM field and $J/\mathbb{F}_p$ is such that $\mathrm{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q} = K$, use isogenies to find the Jacobians whose endomorphism ring is O_K. Work by Lauter+R.

- Modular polynomials in genus 2 using theta null points: computed by Gruenewald using analytic methods for $\ell = 3$.

Question

How to compute $(\ell, 1)$-isogenies in genus 2?

Thank you for your attention!



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