The probability of primality of the order of a genus 2 curve Jacobian

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The genus 1 case: Galbraith-McKee conjecture

- Let $\mathbb{F}_q$ be a finite field of char ≥ 5.
- Let $E : y^2 = x^3 + Ax + B$ be a random elliptic curve.
  - I.e., (A, B) is taken from the set

$$\left\{ (A, B) \in \mathbb{F}_q^2 \;\middle|\; 4A^3 + 27B^2 \neq 0 \right\}$$

    uniformly at random.
- Let $N_E = \#E(\mathbb{F}_q)$.
- Question: what is $P(N_E \text{ is prime})$?
- Motivation: cryptography.

The distribution of N_E

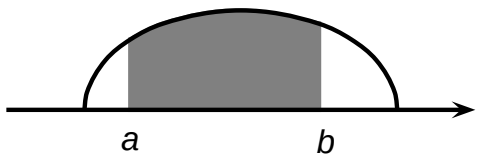
- Hasse's theorem:

$$N_E \in [q+1-2\sqrt{q}, q+1+2\sqrt{q}].$$

- Let's rescale this a bit...
  - Trace of Frobenius: $T_E = q+1-N_E \in [-2\sqrt{q}, 2\sqrt{q}]$.
  - Normalized trace of Frobenius: $t_E = T_E/2\sqrt{q} \in [-1,1]$.

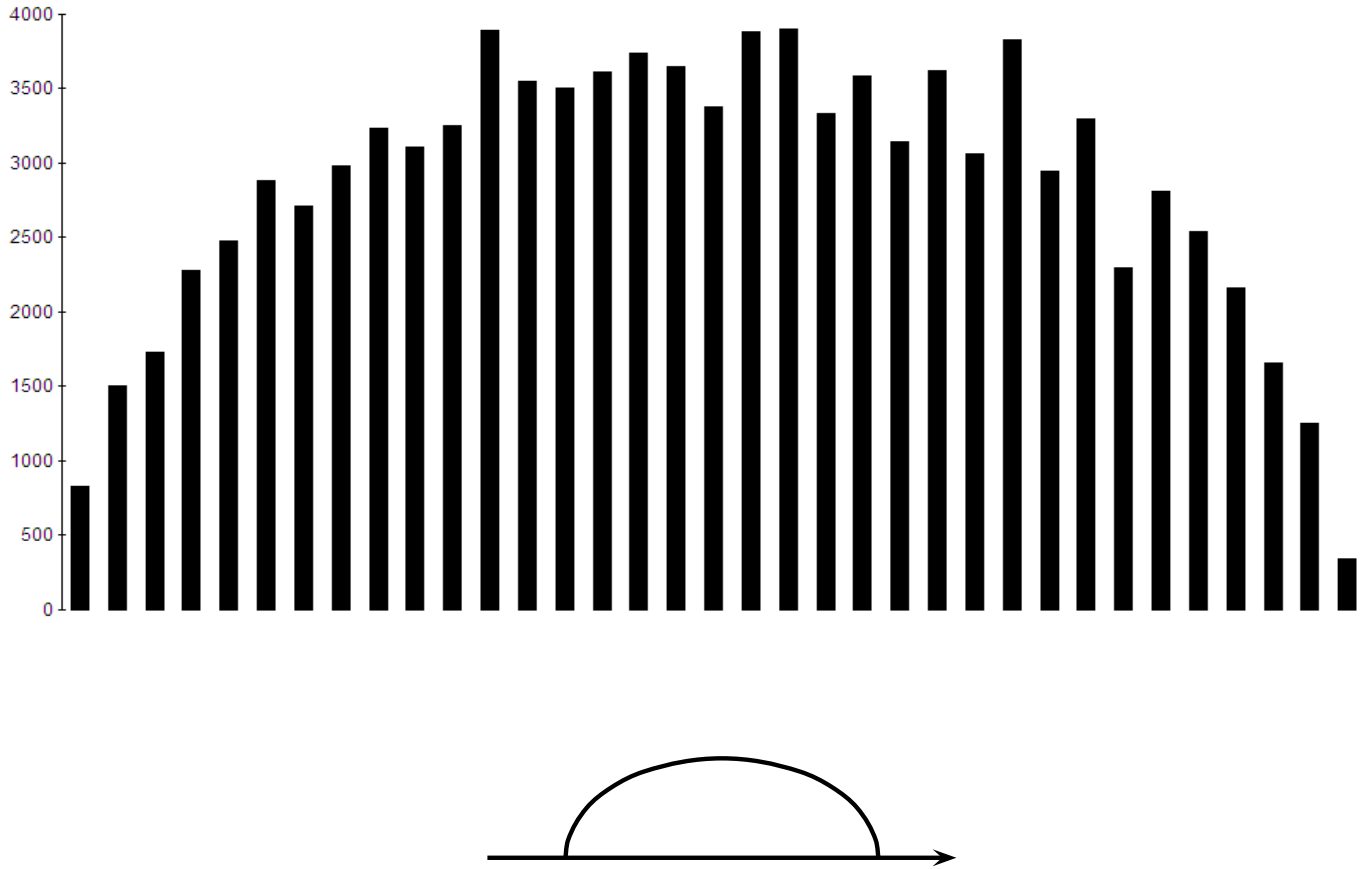
- Birch, Yoshida, Katz-Sarnak: t_E tends to follow a semicircular distribution, i.e.

$$\lim_{q\to\infty} P(a \le t_E \le b) = \int_a^b \frac{2}{\pi}\sqrt{1-t^2}\,dt.$$



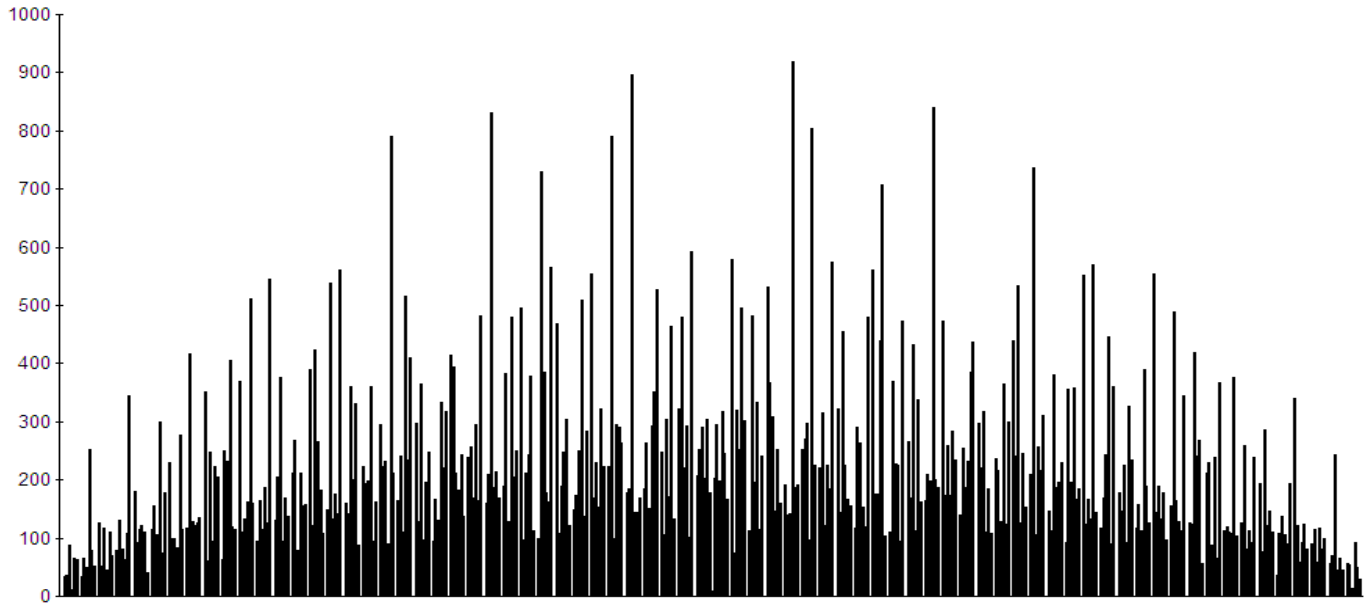
The distribution of N_E

- A histogram of 100.000 curves $y^2 = x^3 + Ax + B$ over $\mathbb{F}_{75}$, with interval width 15.



Subtleties

- The limit dissolves the discrete nature of N_E (or T_E).
- Same experiment, but now interval width 1:



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- Gaps at $T_E \equiv 0 \bmod 7$ (supersingular curves).

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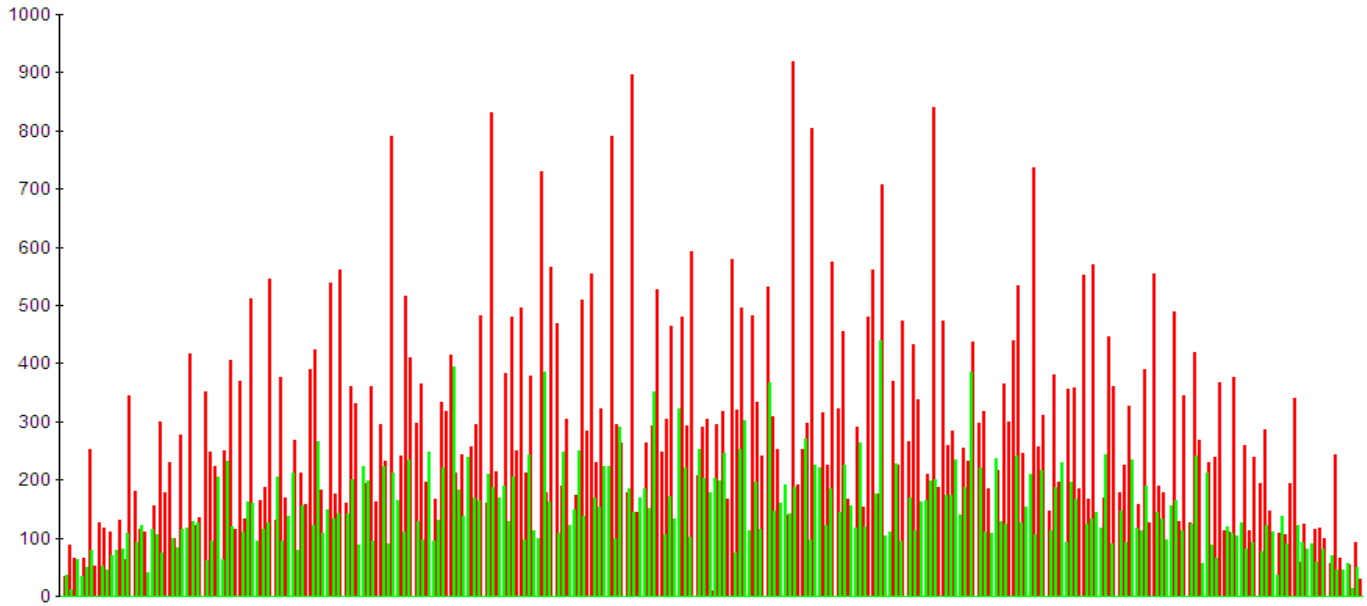

- This doesn't seem to converge to a semicircle very 'smoothly' (lots of peaks and valleys).
- Gaps at $T_E \equiv 0 \bmod 7$ (supersingular curves).

Subtleties

- Easy fact (not very well-known):

$$\lim_{q\to\infty} P(N_E \text{ is even}) = \frac{2}{3}.$$

- Proof:
  - The completing-the-cube map

$$\{\text{square-free } x^3+ax^2+bx+c\} \to \{\text{square-free } x^3+Ax+B\}$$

    is uniform.
  - Thus we may assume that E is defined by $y^2 = f(x)$ for a random square-free $f(x) = x^3+ax^2+bx+c$.
  - N_E is even $\Leftrightarrow$ $E(\mathbb{F}_q)$ has 2-torsion $\Leftrightarrow$ $f(x)$ is reducible.
  - The *irreducible* $f(x)$ are precisely the minimal polynomials of all $\theta \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$, and the correspondence is 3-to-1.
  - Thus

$$\lim_{q\to\infty} P(f(x) \text{ is irreducible}) = \lim_{q\to\infty} \frac{\frac{1}{3}(q^3-q)}{q^3-O(q^2)} = \frac{1}{3}. \qquad \blacksquare$$

Subtleties

- Lenstra: in general, we have

$$\lim_{q\to\infty}\left(P(\ell \mid N_E) - \begin{cases} \frac{1}{\ell-1} & \text{if } q \not\equiv 1 \bmod \ell \\ \frac{\ell}{\ell^2-1} & \text{if } q \equiv 1 \bmod \ell \end{cases}\right) = 0$$

  for any prime number ℓ not dividing q.

- Thus:

$$\ell \ll q \quad \Longrightarrow \quad P(\ell \mid N_E) > \frac{1}{\ell}.$$

  This suggests that $P(N_E$ is prime) is smaller than one would naively expect.

Galbraith-McKee conjecture

- Let's try to quantify this (assume $q = p$ is prime):
- Heuristically (but in fact wrong! $\sim$ Mertens' theorem),

$$P_1(p) = P(\text{random number is prime}) \approx \prod_{\ell \leq \sqrt{p}+1} \frac{\ell - 1}{\ell} \approx \frac{1}{\log p}.$$

- Using Lenstra's estimates, heuristically ('equally wrong'),

$$P_2(p) = P(N_E \text{ is prime}) \approx \prod_{\substack{\ell \nmid p-1 \\ \ell \leq \sqrt{p}+1}} \frac{\ell - 2}{\ell - 1} \cdot \prod_{\substack{\ell \mid p-1 \\ \ell \leq \sqrt{p}+1}} \frac{\ell^2 - \ell - 1}{\ell^2 - 1}.$$

- So:

$$\frac{P_2(p)}{P_1(p)} \approx \frac{\prod_{\ell \nmid p-1} \frac{\ell-2}{\ell-1} \cdot \prod_{\ell \mid p-1} \frac{\ell^2-\ell-1}{\ell^2-1}}{\prod_{\ell} \frac{\ell-1}{\ell}}.$$

Galbraith-McKee conjecture

- Rearranging terms gives:

  Conjecture (Galbraith-McKee, 2000):
  Let

$$c_p = \frac{2}{3} \cdot \prod_{\ell>2} \left(1 - \frac{1}{(\ell-1)^2}\right) \cdot \prod_{\ell \mid p-1,\ \ell>2} \left(1 + \frac{1}{(\ell+1)(\ell-2)}\right),$$

  then

$$\lim_{p\to\infty} (P_2(p)/P_1(p) - c_p) = 0.$$

- $c_p \in [0.44, 0.62]$.
- Galbraith & McKee give a different heuristic argument!
  They use an analytic Hurwitz-Kronecker class number formula counting equivalence classes of bivariate quadratic forms with given discriminant.

Random matrices

- Let $\gcd(n, q) = 1$. To an elliptic curve $E/\mathbb{F}_q$ we can associate its n-torsion subgroup

$$E[n] = \left\{ P \in E\left(\overline{\mathbb{F}}_q\right) \mid nP = \infty \right\}.$$

  It is well-known that

$$E[n] \cong \mathbb{Z}/(n) \times \mathbb{Z}/(n).$$

- Let (P, Q) be a $\mathbb{Z}/(n)$-module basis of $E[n]$, and let $\sigma : E[n] \to E[n]$ be qth power Frobenius. Then we can write

$$P^\sigma = [\alpha]P + [\beta]Q, \quad Q^\sigma = [\gamma]P + [\delta]Q.$$

- Fact: the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in (\mathbb{Z}/(n))^{2\times 2}$$

  has trace $\equiv T_E \bmod n$ and determinant $\equiv q \bmod n$.

Random matrices

- Choosing another basis yields a $\mathrm{GL}_2(\mathbb{Z}/(n))$-conjugated matrix.
- Thus we can unambiguously associate to E a conjugacy class $\mathcal{F}_E$ of matrices of Frobenius (all having trace T_E and determinant q).
- Let $\mathcal{M}_q \subset \mathrm{GL}_2(\mathbb{Z}/(n))$ be the set of matrices of determinant q.

  Quasi-theorem:
  Let $\mathcal{F}$ be a conjugacy class of matrices of determinant q. Then

$$\left| P(\mathcal{F}_E = \mathcal{F}) - \frac{\#\mathcal{F}}{\#\mathcal{M}_q} \right| \leq C \frac{n^2}{\sqrt{q}}.$$

- This is likely to follow from:
  - Chebotarev's density theorem applied to $X(n) \to X(1)$ (in progress)
  - Katz-Sarnak equidistribution as elaborated by Achter, currently modulo some hypotheses.

Example 1

- What proportion of elliptic curves satisfies $E[\ell] \subset E(\mathbb{F}_q)$?
  - $E[\ell] \subset E(\mathbb{F}_q)$ if and only if $E[\ell]$ has a basis consisting of $\mathbb{F}_q$-rational points P and Q.
  - Thus: if and only if
$$\mathcal{F}_E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$
  - By the random matrix theorem, the chance that this happens is
$$\approx \frac{\#\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}}{\#\mathcal{M}_q}.$$
  - $\#\mathcal{M}_q = \ell^3 - \ell$ (exercise).
  - Thus
$$P\left(E[\ell] \subset E(\mathbb{F}_q)\right) \approx \frac{1}{\ell^3 - \ell}.$$

Example 2

- Alternative proof of

$$\lim_{q\to\infty} P(N_E \text{ is even}) = \frac{2}{3}.$$

  - There are 6 elements of $(\mathbb{Z}/(2))^{2\times 2}$ having determinant $q \equiv 1$:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}.$$

  - 4 of them have trace 0.
  - $P(N_E \text{ is even}) = P(q+1-T_E \text{ is even}) = P(T_E \text{ is even}) = 4/6$.

Example 2

- More generally: let ℓ be a prime number not dividing q.
- Exercise:

$$\#\{M \in \mathcal{M}_q \mid q + 1 - \mathrm{Tr}(M) \equiv 0\}$$
$$= \begin{cases} \ell^2 + \ell & \text{if } q \not\equiv 1 \bmod \ell \\ \ell^2 & \text{if } q \equiv 1 \bmod \ell. \end{cases}$$

- Recall: $\#\mathcal{M}_q = \ell^3 - \ell$.
- Hence we recover Lenstra's result:

$$P(\ell \mid N_E) \approx \begin{cases} \frac{\ell^2+\ell}{\ell^3-\ell} = \frac{1}{\ell-1} & \text{if } q \not\equiv 1 \bmod \ell \\ \frac{\ell^2}{\ell^3-\ell} = \frac{\ell}{\ell^2-1} & \text{if } q \equiv 1 \bmod \ell. \end{cases}$$

The genus 2 case

- Let $\mathbb{F}_q$ be a finite field of char ≥ 3.
- Let $H : y^2 = f(x)$ be a random genus 2 curve.
  - I.e., $f(x)$ is taken from either

$$\mathcal{H}_6 = \{ f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, square-free, of degree } 6 \}$$

or

$$\mathcal{H}_5 = \{ f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, square-free, of degree } 5 \}$$

uniformly at random. These are distinct notions!

- Let $N_H = \#\mathrm{Jac}(H)(\mathbb{F}_q)$.
- Question: what is $P(N_H \text{ is prime})$?
- Motivation: cryptography.

Distinct notions of randomness

- What is $P(N_H \text{ is even})$?
- Let $W_1, \ldots, W_6$ be the Weierstrass points of H.
- Every non-zero point of $\mathrm{Jac}(H)[2]$ (thought of as a divisor class) contains a unique pair $\{W_i - W_j, W_j - W_i\}$, where $i \neq j$.
  - Proof: use Riemann-Roch and the fact that there are $\binom{6}{2} = 15$ pairs.
- The pair is $\mathbb{F}_q$-rational iff $\{W_i, W_j\}^\sigma = \{W_i, W_j\}$.
- In case $f(x) \in \mathcal{H}_6$: occurs iff $f(x)$ has a quadratic factor.
  - Exercise: probability $\approx \frac{26}{45}$.
- In case $f(x) \in \mathcal{H}_5$: occurs iff $f(x)$ has a linear or quadratic factor.
  - Exercise: probability $\approx \frac{4}{5}$.

Random matrices in genus 2

- For now, suppose that $f(x)$ is chosen from

$$\mathcal{H}_6 = \{ f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, square-free, of degree } 6 \}$$

uniformly at random.

- This works better from a theoretic point of view.

Random matrices in genus 2

- Let $\gcd(n,q) = 1$. To a genus 2 curve $H/\mathbb{F}_q$ we can associate the n-torsion subgroup of its Jacobian $A = \mathrm{Jac}(H)$:

$$A[n] = \left\{ P \in A\left(\overline{\mathbb{F}}_q\right) \;\middle|\; nP = \infty \right\}.$$

  It is well-known that

$$A[n] \cong (\mathbb{Z}/(n))^4.$$

- Let (P_1, P_2, P_3, P_4) be a $\mathbb{Z}/(n)$-module basis of $A[n]$, then

$$P_1^\sigma = [\alpha_{11}]P_1 + [\alpha_{12}]P_2 + [\alpha_{13}]P_3 + [\alpha_{14}]P_4, \ldots$$

- Fact: the matrix

$$F = \left(\alpha_{ij}\right) \in (\mathbb{Z}/(n))^{4\times 4}$$

  has determinant $\equiv q \bmod n$ and satisfies $\det(F - \mathbb{I}) \equiv N_H \bmod n$.

Random matrices in genus 2

- Choosing another basis yields a $\mathrm{GL}_4(\mathbb{Z}/(n))$-conjugated matrix.
- Thus we can unambiguously associate to H a conjugacy class $\mathcal{F}_H$ of matrices of Frobenius.
- However, a statement like

  Let $\mathcal{M}_q \subset GL_4(\mathbb{Z}/(n))$ be the set of matrices of determinant q. Let $\mathcal{F}$ be a conjugacy class of matrices of determinant q. Then

$$\lim_{q\to\infty}\left(P(\mathcal{F}_H=\mathcal{F})-\frac{\#\mathcal{F}}{\#\mathcal{M}_q}\right)=0.$$

  turns out to be false.

Symplectic structure of $A[n]$

- Let $\zeta_n \in \overline{\mathbb{F}}_q$ be a primitive nth root of unity.
- The Weil pairing
$$e_n : A[n] \times A[n] \to \langle \zeta_n \rangle,$$
when composed with the (non-canonical) map
$$\langle \zeta_n \rangle \to \mathbb{Z}/(n) : \zeta_n^i \mapsto i,$$
is a skew-symmetric, nondegenerate, bilinear pairing $\langle \cdot, \cdot \rangle$ on $A[n]$ (called a symplectic pairing).
- Darboux: $A[n]$ admits a basis with respect to which
$$\langle v, w \rangle = {}^t v \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot w = {}^t v \cdot \Omega \cdot w.$$

Choosing a different root of unity

- Rephrased: a basis $\{P_1, P_2, Q_1, Q_2\}$ is a Darboux basis if

$$e_n(P_i, Q_j) = \zeta_n^{\delta_{ij}}, \quad e_n(P_i, P_j) = e_n(Q_i, Q_j) = 1.$$

- Let $d \in (\mathbb{Z}/(n))^\times$.
- If $\{P_1, P_2, Q_1, Q_2\}$ is a Darboux basis with respect to ζ_n, then $\{P_1, P_2, dQ_1, dQ_2\}$ is a Darboux basis with respect to ζ_n^d.
- Denote the matrix of base change with

$$g_d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

Choosing a different Darboux basis

- Matrix of base change M between two Darboux bases must satisfy

$$ {}^t v \cdot \Omega \cdot w = {}^t(Mv) \cdot \Omega \cdot (Mw) = {}^t v \cdot {}^t M \cdot \Omega \cdot M \cdot w $$

  for all $v, w \in (\mathbb{Z}/(n))^4$.

- Such matrices are called symplectic:

$$ \mathrm{Sp}_4(\mathbb{Z}/(n)) = \left\{ M \in (\mathbb{Z}/(n))^{4\times 4} \;\middle|\; {}^t M \cdot \Omega \cdot M = \Omega \right\}. $$

- Note, if M is symplectic, then

$$ {}^t M \cdot {}^t g_d \cdot \Omega \cdot g_d \cdot M = {}^t g_d \cdot {}^t M \cdot \Omega \cdot M \cdot g_d = d\Omega. $$

Symplectic similitudes

- For $d \in (\mathbb{Z}/(n))^\times$, define the d-symplectic matrices as

$$\mathrm{GSp}_4^{(d)}(\mathbb{Z}/(n)) = \left\{ M \in (\mathbb{Z}/(n))^{4\times 4} \;\middle|\; {}^tM \cdot \Omega \cdot M = d\Omega \right\}.$$

- The symplectic similitudes (generated by $\mathrm{Sp}_4(\mathbb{Z}/(n))$ and $\{g_d\}$):

$$\mathrm{GSp}_4(\mathbb{Z}/(n)) = \bigsqcup_{d\in(\mathbb{Z}/(n))^\times} \mathrm{GSp}_4^{(d)}(\mathbb{Z}/(n)).$$

- Because $e_n(P^\sigma, Q^\sigma) = e_n(P,Q)^q$, the matrix F of σ with respect to a Darboux basis satisfies

$${}^tv \cdot {}^tF \cdot \Omega \cdot F \cdot w = {}^t(Fv) \cdot \Omega \cdot (Fw) = q({}^tv \cdot \Omega \cdot w) = {}^tv \cdot (q\Omega) \cdot w$$

for all $v, w \in (\mathbb{Z}/(n))^4$, i.e., F is q-symplectic.

Random matrices in genus 2 (revisited)

- Thus we can unambiguously associate to H an orbit $\mathcal{F}_H$ of $\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))$ under $\mathrm{GSp}_4(\mathbb{Z}/(n))$-conjugation.
- Quasi-theorem:
  Let $\mathcal{F} \subset \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))$ be an orbit. Then

$$\left| P(\mathcal{F}_H = \mathcal{F}) - \frac{\#\mathcal{F}}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))} \right| \leq C \frac{n^?}{\sqrt{q}}.$$

- This is likely to follow from:
  - A Chebotarev-like statement applied to $\mathcal{A}_2[n] \to \mathcal{A}_2[1]$
  - Katz-Sarnak equidistribution as elaborated by Achter, currently modulo some hypotheses.

Lenstra's theorem in genus 2

- Let ℓ be a prime number not dividing q.
- One can compute

$$\#\left\{ M \in \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(\ell)) \;\middle|\; \det(M - \mathbb{I}) \equiv 0 \right\}$$

$$= \begin{cases} \ell^4(\ell+1)(\ell^2+1)(\ell^2-2) & \text{if } q \not\equiv 1 \bmod \ell \\ \ell^5(\ell^4-\ell-1) & \text{if } q \equiv 1 \bmod \ell. \end{cases}$$

- $\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(\ell)) = \#\mathrm{Sp}_4(\mathbb{Z}/(\ell)) = \ell^4(\ell^4-1)(\ell^2-1)$.
- We conclude:

$$P(\ell \mid N_E) \approx \begin{cases} \frac{\ell^2-2}{(\ell^2-1)(\ell-1)} & \text{if } q \not\equiv 1 \bmod \ell \\ \frac{\ell(\ell^4-\ell-1)}{(\ell^4-1)(\ell^2-1)} & \text{if } q \equiv 1 \bmod \ell. \end{cases}$$

Galbraith-McKee conjecture in genus 2

- Notation:
  - $P_1(p) = P(\text{random number in generalized Hasse interval is prime})$.
  - $P_2(p) = P(N_H \text{ is prime})$.
- Same heuristics yield (suppose again $q = p$ prime):

  Conjecture:
  Let

$$c_p = \frac{38}{45} \cdot \prod_{\ell>2} \left(1 - \frac{1}{(\ell-1)^2} + \frac{1}{(\ell-1)^2} \frac{\ell}{\ell^2-1}\right) \cdot$$

$$\prod_{\ell \mid p-1, \ell>2} \left(1 + \frac{1}{(\ell+1)(\ell-2)} - \frac{1}{(\ell+1)(\ell-2)} \frac{\ell^4 - 2\ell^3 + 2\ell^2 - \ell - 1}{\ell^5 - 2\ell^4 + \ell^2 - \ell + 3}\right),$$

  then

$$\lim_{p \to \infty} (P_2(p)/P_1(p) - c_p) = 0.$$

- Now $c_p \in [0.63, 0.80]$.

Imposing a rational Weierstrass point

- Now, let us briefly discuss the case where $f(x)$ is chosen from

$$\mathcal{H}_5 = \{\, f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, square-free, of degree } 5 \,\}$$

uniformly at random.

- This is often preferred in practice.
- As we've seen, $P(2 \mid N_H)$ increases from $\frac{26}{45}$ to $\frac{4}{5}\ldots$
- What about odd primes ℓ?

Imposing a rational Weierstrass point (skippable)

- Fact: there exist subsets

$$\mathcal{W}_0, \ldots, \mathcal{W}_r \subset \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2))$$

such that

$F \in \mathcal{W}_i$ if and only if H has i rational Weierstrass points.

- The proof uses an isomorphism $\mathrm{Sym}\{W_1, \ldots, W_6\} \cong \mathrm{Sp}_4(\mathbb{Z}/(2))$.

Imposing a rational Weierstrass point (skippable)

- Let $n \nmid q$ be odd and $\mathcal{F} \subset \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))$ be an orbit. Is

$$P(\mathcal{F}_H = \mathcal{F}) \approx \frac{\#\mathcal{F}}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))}?$$

- It suffices to prove this for $f(x)$ randomly chosen from

$$\mathcal{H}_5^{(i)} = \{\, f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, sqf, deg 5, with } i \ \mathbb{F}_q\text{-roots} \,\}$$

for $i = 0, \ldots, 5$, since these partition $\mathcal{H}_5$.

- . . . and even from

$$\mathcal{H}_6^{(i)} = \{\, f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ sqf, deg 6, with } i \ \mathbb{F}_q\text{-roots} \,\}$$

for $i = 1, \ldots, 6$, since the classical swipe-a-point-to-infinity relation $\mathcal{H}_6^{(i)} \to \mathcal{H}_5^{(i-1)}$ is generically uniform.

Imposing a rational Weierstrass point (skippable)

- Thus: estimate $P(\mathcal{F}_H = \mathcal{F} \mid \mathcal{F}_{H,2} \subset \mathcal{W}_i)$ in old $\mathcal{H}_6$-sense.
- Since n is odd, Chinese remaindering gives

$$\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2n)) \cong \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n)) \oplus \mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2)).$$

- We can rewrite

$$P(\mathcal{F}_H = \mathcal{F} \mid \mathcal{F}_{H,2} \subset \mathcal{W}_i) = \frac{P(\mathcal{F}_H = \mathcal{F} \text{ and } \mathcal{F}_{H,2} \subset \mathcal{W}_i)}{P(\mathcal{F}_{H,2} \subset \mathcal{W}_i)} = \frac{P(\mathcal{F}_{H,2n} \subset \mathcal{F} \oplus \mathcal{W}_i)}{P(\mathcal{F}_{H,2} \subset \mathcal{W}_i)}.$$

- By the random matrix statement, we have

$$P(\mathcal{F}_{H,2n} \subset \mathcal{F} \oplus \mathcal{W}_i) \approx \frac{\#(\mathcal{F} \oplus \mathcal{W}_i)}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2n))} = \frac{\#\mathcal{F}}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(n))} \cdot \frac{\#\mathcal{W}_i}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2))}$$

  and

$$P(\mathcal{F}_{H,2} \subset \mathcal{W}_i) \approx \frac{\#\mathcal{W}_i}{\#\mathrm{GSp}_4^{(q)}(\mathbb{Z}/(2))}.$$

- Taking the quotient gives the requested result.

Imposing a rational Weierstrass point

- Taking $f(x)$ from $\mathcal{H}_5$ only affects $P(N_H \text{ is even})$.
- Conclusion: same Galbraith-McKee generalization, with c_p replaced by $\frac{9}{19}c_p \in [0.30, 0.38]$.

Asymptotics for $g \to \infty$

- Let $f(x)$ be chosen from

  $$\mathcal{H}_6 = \{ f(x) \in \mathbb{F}_q[x] \mid f(x) \text{ monic, square-free, of degree } 2g+2 \}$$

  uniformly at random and let $H : y^2 = f(x)$.
- Let $\mathcal{F}_H \subset \mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/(n))$ be the orbit of Frobenius of H.
- Quasi-theorem:
  Let $\mathcal{F} \subset \mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/(n))$ be an orbit. Then

$$\left| P(\mathcal{F}_H = \mathcal{F}) - \frac{\#\mathcal{F}}{\#\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/(n))} \right| \leq C(g) \frac{n^{?(g)}}{\sqrt{q}}.$$

- This is likely to follow from:
  - A Chebotarev-like statement applied to $\mathcal{A}_g[n] \to \mathcal{A}_g[1]$
  - Katz-Sarnak equidistribution as elaborated by Achter, currently modulo some hypotheses.

Asymptotics for $g \to \infty$

- A recursive formula due to Achter-Holden allows us to count

$$\# \left\{ M \in \mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/(\ell)) \;\middle|\; \det(M - \mathbb{I}) \equiv 0 \right\}$$

for $g = 2, 3, 4, 5, 6, \ldots$

- Seemingly, the proportion converges to (and fluctuates around)

$$= \begin{cases} 1 - \phi(1/\ell) & \text{if } q \not\equiv 1 \bmod \ell \\ 1 - \frac{\phi(1/\ell)}{\phi(1/\ell^2)} & \text{if } q \equiv 1 \bmod \ell, \end{cases}$$

where

$$\phi(q) = \prod_{j=1}^{\infty} (1 - q^j)$$

is the Euler q-series.

Asymptotics for $g \to \infty$

- Applying our heuristics gives the following limiting Galbraith-McKee interval:

$$\left[\frac{\phi(1/4)^{-1}}{\prod_{k=2}^{\infty}\zeta(k)}, \frac{1}{\prod_{k=1}^{\infty}\zeta(2k+1)}\right] \approx [0.63287, 0.79353].$$

- Genus 2 is actually the least deviating case!

Concluding remarks

- Generalizable to arbitrary fields (include supersingular cases).
- Generalizable to $P(N_H$ is prime up to a given cofactor$)$.
- Adaptable to $\#H(\mathbb{F}_q)$ instead of $\#\mathrm{Jac}(H)(\mathbb{F}_q)$, but no nice formulas (matrix count involves non-rational varieties).

Concluding remarks

- Thanks for listening!